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PROCEEDINGS ARTICLE

Differential Geometry of Position Vector Fields

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ABSTRACT

Differential geometry studies the geometry of curves, surfaces and higher dimensional smooth manifolds. For submanifolds in Euclidean spaces, the position vector is the most natural geometric object. Position vectors find applications throughout mathematics, engineering and natural sciences. The purpose of this survey article is to present six research topics in differential geometry in which the position vector plays a very important role. In addition to this, we explain the link between position vectors with mechanics, dynamics, and D'Arcy Thompson's law of natural growth in biology.

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1. WHAT IS DIFFERENTIAL GEOMETRY? WHERE IS IT USED?

Differential geometry studies the geometry of curves, surfaces, and higher dimensional smooth manifolds. It uses the ideas and techniques of differential and integral calculus, linear and multilinear algebras, topology, and differential equations. This subject in mathematics is closely related to differential topology, which concerns itself with properties of smooth manifolds. Differential geometry also closely relates to the geometric aspects of the theory of differential equations, otherwise known as geometric analysis.

Curvature is an important notion in mathematics, which has been investigated extensively in differential geometry. There are two types of curvatures: namely, "intrinsic" and "extrinsic".

"Intrinsic curvature" describes the curvature at a point on a surface or a smooth manifold and is independent of how the surface or manifold is embedded in space. Borrow a term from biology, intrinsic invariants of a manifold are the DNA of the manifold. The Gauss curvature of a surface is the most commonly studied intrinsic measure of curvature. In higher dimensions, curvature is too complicated to be described by a single number. In this case, tensors are used to describe the curvature as pioneered by B. Riemann in his famous 1854 inaugural lecture at Gottingen:

> "*Über die Hypothesen welche der Geometrie zu Grunde liegen.*"

In Einstein's theory of general relativity, intrinsic curvature is key to understanding the shape of the universe.

"Extrinsic curvature" of a manifold depends on how it is embedded within a space. Examples of extrinsic measures of curvature include geodesic curvature, principal curvature, and mean curvature. The most important extrinsic invariant for a submanifolds in an ambient Riemannian manifold is the mean curvature vector, which is known as the tension field in physics.

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Differential geometry has numerous applications in mathematics and natural sciences. Most prominently, Albert Einstein used differential geometry for his theory of general relativity. More recently, differential geometry was applied by physicists in the development of quantum field theory and the standard model of particle physics. Outside of physics, differential geometry finds many applications in botany, biology, economics, chemistry, engineering, medical imaging, control theory, computer graphics and vision, and recently in machine learning.

2. BASIC NOTATIONS AND FORMULAS

For the general references in this section, we refer to [1,2,3,4,5,6,7].

Let $\mathbf{x}\colon M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into the Euclidean m-space $\mathbb{E}^m$. For each point $p \in M$, we denote by T_pM and $T_p^{\perp}M$ the tangent and the normal spaces at p. There is a natural orthogonal decomposition:

$$T_p\mathbb{E}^m = T_pM \oplus T_p^{\perp}M$$

Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and $\mathbb{E}^m$, respectively. Then the formulas of Gauss and Weingarten are given respectively by:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connection, and A the shape operator of M.

For a normal vector ξ at p, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A are related by:

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle \tag{2.3}$$

where $\langle , \rangle$ is the inner product on M as well as on the ambient Euclidean space. The mean curvature vector of M is defined by:

$$H = \frac{1}{n}\ \text{Trace}\ h \tag{2.4}$$

where $n = \dim\ M$. At a given point $p \in M$, the *first normal space* of M in $\mathbb{E}^m$, denoted by $\text{Im}\ h_p$, is the subspace defined by:

$$\text{Im}\ h_p = \text{Span}\left\{h(X,Y)\colon X, Y \in T_pM\right\} \tag{2.5}$$

A submanifold M is called *totally geodesic* if its second fundamental form h vanishes identically; and *totally umbilical* if for any normal vector field ξ of M, $A_\xi = f_\xi I$ holds for some function f_ξ, where I is the identity map. A submanifold M is called *pseudo-umbilical* if it satisfies $A_H = fI$ for some function f on M. Further, a hypersurface of dimension n is called *quasi-umbilical* if its shape operator has an eigenvalue of multiplicity $\geq n-1$.

For a Riemannian manifold (M, g) with Riemannian connection ∇, the Riemann curvature tensor R is defined by:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.6}$$

for vector fields X, Y, Z tangent to M. The *Ricci curvature tensor* of M, denoted by Ric, is defined by:

$$Ric\ (X,Y) = \text{trace}\ \{Z \mapsto R(Z,X)Y\} \tag{2.7}$$

The *scalar curvature* ρ is defined as the trace of Ricci curvature tensor. For surfaces, the scalar curvature is nothing but the Gaussian curvature.

A Riemannian manifold of dimension $n \geq 3$ is called an *Einstein manifold* if it satisfies $Ric = cg$ for a constant c.

For a submanifold M lying in a Euclidean space, the most elementary and natural geometric object is the position vector $\mathbf{x}$ of M. The position vector, also known as location vector or radius vector, is a Euclidean vector $\mathbf{x} = \overrightarrow{op}$ that represents the position of a point $p \in M$ in relation to an arbitrary reference origin o.

Among extrinsic invariants of a submanifold, the most natural and important one is the mean curvature vector H. In physics, the mean curvature vector is the tension field imposed on the submanifold arising from the ambient space. It is well-known that the surface tension is responsible for the shape of liquid droplets. In materials science, surface tension is used for either surface stress or surface free energy.

The position vector field $\mathbf{x}$ and the mean curvature vector H of M is linked by the well-known formula of E. Beltrami:

$$\Delta\mathbf{x} = -nH \tag{2.8}$$

where Δ is the Laplacian of M with respect to its induced metric on M from the metric of the ambient space.

If the mean curvature vector vanishes identically on a submanifold M, then it is called a *minimal submanifold*. The history of minimal submanifolds goes back to J.L. Lagrange who initiated the study of minimal surfaces in $\mathbb{E}^3$ in [8]. Since then, the theory of minimal surfaces has attracted many mathematicians for more than two centuries. In particular, minimal surfaces and minimal submanifolds in Riemannian manifolds of constant curvature have been studied extensively (see e.g. [9,10,11,12,13]).

The position vector plays important roles in physics, especially in mechanics. In any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most soughtafter quantity because it defines the motion of a particle – its location at some time variable t. It is well-known that the first and the second derivatives of the position vector with respect to time t gives rise to the velocity and acceleration of the particle.

There are many beautiful links between geometry and botany, biology, etc. as already mentioned in several talks delivered at this Symposium on "**Square Bamboos and the Geometree**" held on 21-22 November 2022, as well as illustrated in J. Gielis' book "*The Geometrical Beauty of Plants*" [14]. The purpose of this survey article is thus to present six research topics in differential geometry in which the position vector plays a very important role.

3. TOPIC I: THOMPSON'S LAW OF NATURAL GROWTH AND DIFFERENTIAL GEOMETRY

D'Arcy Thompson was a pioneer of mathematical biology. He was elected a Fellow of the Royal Society, was knighted, and received the Linnean Medal (1938) and the Darwin Medal (1946) for his important contribution in biology. His most famous work is his book *"On Growth and Form"* [15] originally published in 1917 with many revised editions (Fig. 1). Thompson's theory of growth and form provided an excellent link between biology and differential geometry of position vector fields.

The central theme of Thompson's book is that biologists of his time overemphasized evolution as the fundamental determinant of the form and structure of living organisms and underemphasized the roles of physical laws and mechanics. Therefore, he advocated structuralism as an alternative to survival of the fittest in governing the form of species.

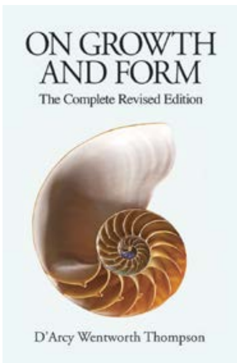

Figure 1. Cover image of the book "*On Growth and Form*" by D'Arcy Wentworth Thompson.

On the concept of "allometry" of his study of the relationship of body size and shape, Thompson wrote: *"An organism is so complex a thing, and growth so complex a phenomenon, that for growth to be so uniform and constant in all the parts as to keep the whole shape unchanged would indeed be an unlikely and an unusual circumstance. Rates vary, proportions change, and the whole configuration alters accordingly."*

In the section "The Equiangular Spiral in Its Dynamical Aspect", he wrote: *"In mechanical structures, curvature is essentially a mechanical phenomenon. It is found in flexible structures as a result of bending, or it may be introduced into construction for the purpose of resisting such a bending-moment. But neither shell nor tooth nor claw are flexible structures; they have not been bent into their peculiar curvature, they have grown into it.*

We may for a moment, however, regard the equiangular or logarithmic spiral of our shell from the dynamic point of view, by looking at growth itself as the force concerned. In the growing structure, let growth at a point P be resolved into a force F acting along the line joining P to a pole O, and a force T acting in a direction perpendicular to OP; and let the magnitude of these forces (or of these rates of growth) remain constant. It follows that the resultant of the forces F and T (as PQ) makes a constant angle with the radius vector [position vector]. But a constant angle between tangent and radius vector [position vector] is a fundamental property of the "equiangular" spiral: the very property with which Descartes started his investigation, and that which gives its alternative name to the curve.

In such a spiral, radial growth and growth in the direction of the curve bear a constant ratio to one another. For, if we consider a consecutive radius vector OP', whose increment as compared with OP is dr, while ds is the small arc PP', then $dr/ds = \cos\ \alpha = constant$.

In the growth of a shell, we can conceive no simpler law than this, that it shall widen and lengthen in the same unvarying proportions: and this simplest of laws is that which Nature tends to follow. The shell, like the creature within it, grows in size but does not change its shape; and the existence of this constant relativity of growth, or constant similarity of form, is of the essence, and may be made the basis of a definition, of the equiangular spiral."

Thompson's law of natural growth has a natural link[1] to the author's *constant-ratio submanifolds* in his study of the position vector published in [16].

Let $\mathbf{x}$ denote the position vector of a submanifold M of $\mathbb{E}^m$. Then there exists an orthogonal decomposition:

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^\perp \tag{3.1}$$

where $\mathbf{x}^T$ and $\mathbf{x}^\perp$ are the tangential and normal components of $\mathbf{x}$, respectively. Let $\|\mathbf{x}^T\|$ and $\|\mathbf{x}^\perp\|$ denote the length of $\mathbf{x}^T$ and $\mathbf{x}^\perp$, respectively. Then a submanifold M of $\mathbb{E}^m$ is said to be of *constant-ratio* if the ratio $\|\mathbf{x}^T\|$: $\|\mathbf{x}^\perp\|$ is a constant. A Euclidean submanifold M is called a *proper submanifold* if both $\mathbf{x}^T$ and $\mathbf{x}^N$ are nowhere zero on an open dense subset of M.

Note that a constant-ratio curve in a plane is exactly an equiangular curve in the sense of Thompson. Hence, constant-ratio submanifolds can be regarded as a higher dimensional version of Thompson's equiangular curves. For this reason, constant-ratio submanifolds are also known in some literature as *equiangular submanifolds* (see e.g. [17,18,19]).

Constant-ratio submanifolds in Euclidean spaces and space-like constant ratio submanifolds in pseudo-Euclidean spaces have been completely classified in [16,20].

Remark 3.1. Constant ratio submanifolds are also related to the notion of *convolution manifolds* introduced by the author in [21,22].

Remark 3.2. It was known in [16] that the tangential component $\mathbf{x}^T$ of the position vector field $\mathbf{x}$ of a constant-ratio hypersurface in $\mathbb{E}^{n+1}$ defines a principal direction. In [23] Y. Fu and M.I. Munteanu called a surface in $\mathbb{E}^3$ a *generalized constant-ratio surface* if $\mathbf{x}^T$ is a principal direction. They proved in [23] that a generalized constant-ratio surface in $\mathbb{E}^3$ can be parametrized as:

$$x(s,t) = s(\cos\ u(s)\gamma(t) + \sin\ u(s)\gamma(t) \times \gamma'(t))$$

where γ is a unit speed curve on the unit 2-sphere centered at the origin and $u(s) = \int_0^s t^{-1} \cot\theta(t)\, dt$ for some function $\theta(s) \in \left(0, \frac{\pi}{2}\right)$.

[1] The author thanks Leopold Verstraelen who pointed out this nice link to the author several years after the appearance of [16].

4. TOPIC II: RECTIFYING CURVES AND RECTIFYING SUBMANIFOLDS

In elementary differential geometry, most geometers describe a curve as a unit speed curve $\mathbf{x} = \mathbf{x}(s)$ whose position vector is expressed in term of an arc-length parameter s. In order to define the curvature and torsion for space curves, we need to recall the Frenet-Serret formulas of space curves which are defined as follows.

Let $\mathbf{x}: I \to \mathbb{E}^3$ be a unit-speed smooth curve defined on an open interval $I = (\alpha, \beta)$. Let us put $\mathbf{t} = \mathbf{x}'(s)$. It is possible that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can define a unique vector field $\mathbf{n}$ and a positive function κ in such way that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{n}$ the *principal normal vector* and κ the *curvature* of the curve. Since $\mathbf{t}$ is of constant length, $\mathbf{n}$ must be perpendicular to $\mathbf{t}$. The *binormal vector* is then defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, which is a unit vector field perpendicular to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$. A curve in $\mathbb{E}^3$ is called *twisted* if has non-zero curvature and non-zero torsion.

The famous *Frenet-Serret formulas* are given by:

$$\begin{cases} \mathbf{t}' = \kappa\mathbf{n} \\ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' = -\tau\mathbf{n} \end{cases} \tag{4.1}$$

At each point of the curve, the three basic planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are called the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively.

A *helix* is a curve in $\mathbb{E}^3$ which satisfies the property that its tangents make a constant angle with a fixed line L, called the axis. It is known in classical differential geometry that a curve in $\mathbb{E}^3$ is a planar curve if and only if its position vector lies in its osculating plane at each point; and a curve lies in a sphere if and only if its position vector always lies in its normal plane. In view of these two basic facts, the author asked in [24] the following simple and natural geometric question:

Question. *When does the position vector of a space curve $\boldsymbol{x}: I \to \mathbb{E}^3$ always lie in its rectifying plane?*

The author simply called such a curve a rectifying curve. Obviously, the position vector of a rectifying curve satisfies:

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{4.2}$$

for some functions λ and μ.

4.1. Physical Interpretation of Rectifying Curves

If a moving point traverses a curve in such a way that s is the time parameter, then the Frenet-Serret frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ moves in accordance with the Frenet-Serret formulas in Eq. (4.1). It is well known in mechanics that this motion contains, apart from an instantaneous translation, an instantaneous rotation with angular velocity vector given by the following *Darboux's rotation vector:*

$$\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$$

The direction of the Darboux rotation vector is that of the *instantaneous axis of rotation* and its length $\sqrt{\kappa^2 + \tau^2}$ is called the angular speed (see e.g. [25]). By applying Eq. (4.2), we can prove that *rectifying curves are exactly the space curves whose axis of instantaneous rotation always passes through the origin of* $\mathbb{E}^3$.

4.2. Comparison of Helices and Rectifying Curves

A well-known theorem of M.A. Lancret [26] proven in 1806 stated that a curve in $\mathbb{E}^3$ is a helix if and only if the ratio $\tau: \kappa$ is a non-zero constant.

For rectifying curves, we have the following result from [24]:

Theorem 4.1. *A curve* $\mathrm{x}: I \to \mathbb{E}^3$ *is a rectifying curve if and only if the ratio $\tau: \kappa$ is a non-constant linear function in an arc-length function s.*

The fundamental theorem for space curves in $\mathbb{E}^3$ states that, up to rigid motions, a curve is uniquely determined by its curvature and torsion given as functions of the arc-length. It invokes solving the Frenet-Serret equations in order to determine the space curves.

On the other hand, a result of S. Lie and J.-G. Darboux showed that solving the Frenet-Serret equations is equivalent to solving the following complex Riccati equation:

$$\frac{dw}{ds} = \mathrm{i}\left(\frac{\tau}{2}w^2 - \frac{\tau}{2} - \kappa w\right) \tag{4.3}$$

In practice, for a space curve with prescribed curvature κ and torsion τ, the solutions of differential equation Eq. (4.3) are often impossible to find explicitly. Fortunately, the author was able to determine all rectifying curves in $\mathbb{E}^3$ explicitly in [24] as follows:

Theorem 4.2. *A curve in $\mathbb{E}^3$ is a rectifying curve if and only if it is given by:*

$$\mathbf{x}(t) = a\sec(t + b)\mathbf{y}(t) \tag{4.4}$$

where $a \neq 0$ and b are real numbers, and $\mathbf{y} = \mathbf{y}(t)$ is a unit-speed curve in the unit sphere $S^2(1) \subset \mathbb{E}^3$ centered at the origin o.

For a unit speed curve $\mathbf{y} = \mathbf{y}(t)$ lying on $S^2(1) \subset \mathbb{E}^3$, let $C\mathbf{y}$ denote the *cone with vertex at $o \in \mathbb{E}^3$ over the spherical curve $\mathbf{y}$*. We may parametrize $C_{\mathbf{y}}$ as:

$$C_{\mathbf{y}}(t, u) = u\mathbf{y}(t) \qquad u \in \mathbf{R}^+ \tag{4.5}$$

A well-known result in classical differential geometry states that *a helix is a geodesic on the cylinder in $\mathbb{E}^3$ containing the helix*.

On the other hand, for rectifying curves we have the following result from [27]:

Theorem 4.3. *A rectifying curve γ is a geodesic on the cone $\mathrm{C}_{\mathbf{y}}$ containing γ, where $\mathbf{y}$ is defined by Eq. (4.4).*

Besides those results mentioned above, rectifying curves have many other nice properties (see e.g. [24,27,28,29]). During the last two decades, there are many articles investigating rectifying curves in various ambient spaces and many new results in this respect have been obtained (see e.g. [30,31,32,33,34]).

4.3. Rectifying Submanifolds

For a curve $\mathbf{x}: I \to \mathbb{E}^3$ with $\kappa(s_0) \neq 0$ at $s_0 \in I$, the first normal space of the curve at s_0 is the line spanned by the principal normal $\mathbf{n}(s_0)$. Therefore, the rectifying plane at s_0 is exactly the plane orthogonal to the first normal space at s_0. Consequently, for a submanifold $M \subset \mathbb{E}^m$ and a point $p \in M$, we may call the subspace of $T_p\mathbb{E}^m$ which is the orthogonal complement to the first normal space Im σ_p to be the *rectifying space* of M at p. Based on this simple fact, the author extended the notion of rectifying curves to rectifying submanifolds in [35] as follows.

A submanifold M of a Euclidean space is called a *rectifying submanifold* if the position vector field $\mathbf{x}$ of M always lies in its rectifying space. In other words, M is called a rectifying submanifold if and only if $\mathbf{x}(p) \perp \mathrm{Im}\ h_p$ at every point $p \in M$.

5. TOPIC III: FINITE TYPE SUBMANIFOLDS

The notion of finite type submanifolds was introduced around the beginning of 1980s via the author's attempts to find the best possible estimates of the total mean curvature for compact Euclidean submanifolds, and also in the late 1970s to search for a notion of "degree" for general submanifolds in Euclidean spaces (see [1,36,37]). This topic of *finite type* submanifolds is another active research topic in which the position vectors of Euclidean submanifolds play important roles.

Let Δ denote the Laplacian of a submanifold M in $\mathbb{E}^m$ as before. Then M is said to be of *finite type* if its position vector $\mathbf{x}$ admits a finite spectral decomposition with respect to Δ:

$$\mathbf{x} = c + \mathbf{x}_1 + \cdots + \mathbf{x}_k \tag{5.1}$$

where c is a constant vector and $\mathbf{x}_1, \dots, \mathbf{x}_k$ are non-constant maps satisfying:

$$\Delta \mathbf{x}_i = \lambda_i \mathbf{x}_i \qquad i = 1, \dots, k \tag{5.2}$$

for some eigenvalues $\lambda_1, \dots, \lambda_k$ of Δ. If $\lambda_1, \dots, \lambda_k$ in Eq. (5.2) are mutually different, then the submanifold M is said to be of k-type. A submanifold of a Euclidean space is said to be of *infinite type* if it is not of finite type.

The family of submanifolds of finite type is very large since it contains many important families of submanifolds, e.g. all minimal submanifolds of Euclidean spaces, all minimal submanifolds of hyperspheres, all parallel submanifolds, and all equivariantly immersed compact homogeneous submanifolds in Euclidean spaces. Just like minimal submanifolds, submanifolds of finite type are characterized by a spectral variation principle; namely, as critical points of directional deformations (see [38] for details).

On one hand, the study of finite type submanifolds provides a natural way to link spectral geometry with the theory of submanifolds. On the other hand, we can apply the theory of finite type submanifolds to obtain some important information on the spectral geometry of submanifolds.

The first results on finite type submanifolds as well as results on finite type maps were collected in author's books [1,39] published in the middle of the 1980s. Further, a list of twelve open problems and three conjectures on finite type submanifolds was published in 1991 (see [40]). Furthermore, a comprehensive survey of results on this topic up to 1996 was given in [41]. Moreover, an up-to-date comprehensive survey, up to 2015, on this topic was presented in the author's book [42]. For more results on this, we refer to [40,41,42,43,44].

Two main conjectures on finite type submanifolds are the following (see [39,41]):

Conjecture A. *The only compact hypersurfaces of finite type in Euclidean space are ordinary hyperspheres*.

Conjecture B. *The only finite type surfaces in $\mathbb{E}^3$ are minimal surfaces, open portions of spheres, and open portions of circular cylinders.*

Although there are many articles which provide affirmative partial answers to support these two conjectures, both of them remain open since 1985.

6. TOPIC IV: BIHARMONIC SUBMANIFOLDS AND BIHARMONIC CONJECTURES

According to Beltrami's formula in Eq. (2.8), a submanifold of a Euclidean space is a minimal submanifold if and only if its position vector $\mathbf{x}$ is harmonic, i.e. $\Delta\mathbf{x} = 0$. Therefore, a submanifold $M \subset \mathbb{E}^m$ is called a *biharmonic submanifold* if its position vector field satisfies the following biharmonic condition:

$$\Delta^2 \mathbf{x} = 0 \tag{6.1}$$

Obviously, every minimal submanifold of a Euclidean space is always biharmonic. Hence, the real question on biharmonic submanifolds is:

"When a biharmonic submanifold is minimal or harmonic?"

It follows easily from Hopf's lemma and Eq. (6.1) that every biharmonic submanifold of a Euclidean space is non-compact. The study of biharmonic submanifolds in Euclidean spaces was raised by the author in the middle of the 1980s via his program in understanding finite type submanifolds (and independently by G.-Y. Jiang [45] in his study of the Euler-Lagrange's equation of bi-energy functional via Eells-Sampson's work [46]).

The author showed in the middle of 1980s that biharmonic surfaces in $\mathbb{E}^3$ are always minimal (unpublished then). This result was the starting point of I. Dimitrić's work on his doctoral thesis [47] at Michigan State University. In this respect, Dimitrić extended the author's result on biharmonic surfaces in $\mathbb{E}^3$ to biharmonic hypersurfaces of $\mathbb{E}^{n+1}, n \geq 3$, with at most two distinct principal curvatures in [47]. Moreover, Dimitrić proved that every biharmonic submanifold of finite type in Euclidean space is always minimal, regardless of codimension. He also proved that every pseudo-umbilical biharmonic submanifold of a Euclidean space is always minimal.

About 30 years ago, the author made the following biharmonic conjecture:

Conjecture. *The only biharmonic submanifolds of Euclidean space are the minimal ones* [40].

There are many articles published during the last 30 years to support this biharmonic conjecture (see e.g. [18,47,48,49,50,51,52,53,54,55]). However, this conjecture remains open until now.

The next conjecture was made by R. Caddeo, S. Montaldo and C. Oniciuc in [56,57] which can be regarded as an extension of the author's biharmonic conjecture:

Generalized Chen's Conjecture. *Every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.*

In [58], Y.-L. Ou and L. Tang proved that Generalized Chen's Conjecture is false in general, by constructing foliations of proper biharmonic hyperplanes in some conformally flat 5-manifolds with negative sectional curvature (see also [59]). On the other hand, there are many results since the early 2000s which support the Generalized Chen's Conjecture under some additional conditions on the ambient spaces (see e.g. [56,60,61,62,63,64,65] among many others).

Nowadays, the study of biharmonic submanifolds is a very active research topic. Biharmonic submanifolds have received growing attention with much progress made since the beginning of this century.

For a comprehensive survey of results on biharmonic submanifolds and on biharmonic maps up to 2020, we refer to the book [7] by Y.-L. Ou and the author, and also to the references mentioned in [7].

7. TOPIC V: MEAN CURVATURE FLOWS AND SELF-SHRINKERS

In differential geometry, a *geometric flow*, or a *geometric evolution equation*, is a type of geometric object such as a Riemannian metric or an embedding. The most well-known geometric flows are *mean curvature flows*, *Ricci flows* and *Yamabe flows*.

An important class of solutions of mean curvature flows is the class of self-shrinkers. And the most important families of solutions for Ricci flows and Yamabe flows are "Ricci solitons" and "Yamabe solitons", respectively.

A *mean curvature flow* of an immersion $\mathbf{x}: M \to \mathbb{E}^m$ is a one-parameter family $\mathbf{x}_t = \mathbf{x}(\cdot, t)$ of immersions $\mathbf{x}_t: M \to \mathbb{E}^m$ such that:

$$\frac{\partial}{\partial t}\mathbf{x}(p,t) = H(p,t), \quad \mathbf{x}(p,0) = \mathbf{x}(p) \qquad p \in M \tag{7.1}$$

is satisfied, where $H(p,t)$ is the mean curvature vector of $M_t \subset \mathbb{E}^m$ at $\mathbf{x}(p,t)$. Thus, the variational vector field of the mean curvature flow is the mean curvature vector.

The most well-known example of mean curvature flow is the evolution of soap films. Intuitively, a family of submanifolds evolves under mean curvature flow if the normal component of the velocity of which a point on the submanifolds moves is given by the mean curvature vector. The mean curvature flow of a surface extremalizes surface area. Further, minimal surfaces are the critical points for the mean curvature flow.

A submanifold $M \subset \mathbb{E}^m$ is called a *self-shrinker* if it satisfies the following quasilinear elliptic system:

$$H = -\mathbf{x}^{\perp} \tag{7.2}$$

Self-shrinkers have the property that their evolution under the action of the mean curvature flow is a shrinking homothety. The study of self-shrinkers is important since the blow-up of the mean curvature flow at a singularity, under certain assumptions, is self-shrinking.

Now let us mention the following known results on self-shrinkers:

(1) U. Abresch and J. Langer classified in [66] self-shrinker closed curves in $\mathbb{E}^2$. They proved that circles are the only imbedded self-shrinkers in $\mathbb{E}^2$.

(2) G. Huisken studied in [67] compact self-shrinkers, and proved that if a compact self-shrinker hypersurface in $\mathbb{E}^{n+1}$ has non-negative mean curvature, then it is a hypersphere of $\mathbb{E}^{n+1}$ with radius $\sqrt{n}$.

(3) Compact imbedded self-shrinker $S^1 \times S^{n-1}(\sqrt{n-1}) \subset \mathbb{E}^{n+1}$ was constructed by S. B. Angenent in [68].

(4) A. Kleene and N.M. Moller proved in [69] that a complete imbedded self-shrinking hypersurface of revolution in $\mathbb{E}^{n+1}$ is isometric to $\mathbb{E}^n, S^n(\sqrt{n}), \mathbf{R} \times S^{n-1}(\sqrt{n-1})$, or $S^1 \times S^{n-1}(\sqrt{n-1})$.

(5) N.Q. Le and N. Sesum proved in [70] that if M is a complete embedded selfshrinker hypersurface in $\mathbb{E}^{n+1}$ with polynomial volume growth and $\| h \| < 1$, then $h = 0$, where h denotes the second fundamental form; thus M is isometric to the hyperplane.

In recent years, there are many articles studying self-shrinkers with arbitrary codimension (see for example [45,67,70,71,72,73,74,75,76,77,78,79,80,81]). Nowadays, the study of self-shrinkers is quite an active research topic and much more remains to be done.

8. TOPIC VI: DIFFERENTIAL GEOMETRY OF CANONICAL VECTOR FIELDS

In Topic V, we discussed self-shrinkers which involve the normal component $\mathbf{x}^\perp$ of the position vector field $\mathbf{x}$ of a submanifold $M \subset \mathbb{E}^m$. In this section, we discuss the case in which the tangent component $\mathbf{x}^T$ plays an important role. Obviously, $\mathbf{x}^T$ is the most natural vector field tangent to M, which is called the *canonical vector field of* M.

8.1. Differential Geometry of Canonical Vector Fields

We present some known results on Euclidean submanifolds whose canonical vector fields are of special types.

Recall that a vector field on a Riemannian manifold is called *conservative* if it is the gradient of a scalar function. Such vector fields appear naturally in mechanics. Conservative vector fields have the important property that the line integral is path independent. They represent forces of physical systems in which energy is conserved.

A vector field on a Riemannian manifold is called *incompressible* if it is a vector field with divergence zero at all points in the field. An important family of incompressible vector fields are magnetic fields. Magnetic fields are widely used in modern technology, particularly in electrical engineering and electromechanics (see e.g. [71]).

Concerning conservative and incompressible vector fields, we have the following two results from [82]:

Theorem 8.1. *Let M be a submanifold of $\mathbb{E}^m$. Then:*

(1) The canonical vector field of M is always conservative.

(2) The canonical vector field of M is incompressible if and only if position vector and mean curvature vector of M satisfy $\langle H, \boldsymbol{x} \rangle = -1$ identically.

As an application of this theorem, we have the following result:

Theorem 8.2. *Every equivariantly isometrical immersion of a compact homogeneous Riemannian manifold into a Euclidean space has an incompressible canonical vector field.*

For further results on submanifolds with incompressible canonical vector fields, see [83]. A vector field v on a Riemannian manifold M is called a *conformal vector field* if it satisfies:

$$\mathcal{L}_v g = 2\varphi g \tag{8.1}$$

where $\mathcal{L}$ is the Lie derivative and φ is a scalar function called the *potential function*.

The next two results from [31] characterize Euclidean submanifolds with a conformal canonical vector field:

Theorem 8.3. *Let M be a submanifold of a Euclidean space. Then the canonical vector field of M is a conformal vector field if and only if M is umbilical with respect to the normal component $\boldsymbol{x}^N$ of the position vector $\boldsymbol{x}$.*

For hypersurfaces, we have the following:

Corollary 8.1. *Let M be a proper hypersurface of $\mathbb{E}^{n+1}$ with conformal canonical vector field. Then either:*

(1) M lies in a hypersphere centered at the origin of $\mathbb{E}^{n+1}$; or

(2) M lies in a hyperplane which does not contain the origin of $\mathbb{E}^{n+1}$.

A non-trivial vector field v on a Riemannian manifold (M, g) is called a *concircular vector field* if and only if it satisfies:

$$\nabla_X v = \varphi X \tag{8.2}$$

for some scalar function φ. In particular, if $\varphi = 1$, then v is called a *concurrent vector field*.

The following result was proven by the author and S.W. Wei in [84]:

Theorem 8.4. *Let M be a submanifold of a Euclidean space. Then the canonical vector field of M is conformal if and only if it is concircular.*

The next result characterizes rectifying submanifolds via a canonical vector field:

Theorem 8.5. *If M is a proper submanifold of Euclidean space, then the canonical vector field of M is a concurrent vector field if and only if M is a proper rectifying submanifold* [27,35].

According to K. Yano [85], a vector field v on a Riemannian manifold M is called *torse-forming* if it satisfies:

$$\nabla_X v = \varphi X + \alpha(X) v \qquad \forall X \in TM \tag{8.3}$$

for any vector X tangent to M, where φ is a scalar function and α is a 1-form on M. A torse-forming vector field v is called *proper torse-forming* if the 1-form α in Eq. (8.3) is nowhere zero on a dense open subset of M. A *torqued vector field* is a torse-forming vector field v satisfying Eq. (8.3) with $\alpha(v) = 0$ (see [86,87,88]).

In [89], the author and L. Verstraelen provide a link between hypersurfaces with a torse-forming canonical vector field and rotational hypersurfaces. More precisely, they proved the following:

Theorem 8.6. *Let M be a hypersurface of $\mathbb{E}^{n+1}$ with $n \geq 3$. Then the canonical vector field of M is a proper torse-forming vector field if and only if M is contained in a rotational hypersurface whose axis of rotation contains the origin.*

For further results in this direction, see the survey articles [90,91].

8.2. Ricci Solitons With Canonical Vector Fields as Soliton Fields

Ricci flows and Ricci solitons were introduced by R. Hamilton in the 1980s. A vector field η on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies the Ricci soliton equation:

$$\frac{1}{2}\mathcal{L}_\eta g + \text{Ric} = \lambda g \tag{8.4}$$

where *Ric* is the Ricci tensor and λ is a constant (see e.g. [92,93,94]). The vector field η in Eq. (8.4) is called the *potential field.* We denote such a Ricci soliton by (M, g, η, λ). A Ricci soliton (M, g, η, λ) with dim $M \geq 3$ is called trivial if the Riemannian manifold (M, g) is an Einstein manifold.

The next result was obtained in [88]:

Theorem 8.7. *Let (M, g, η, λ) be a Ricci soliton whose potential field η is a torqued vector field. Then (M, g, η, λ) is trivial if and only if η is a concircular vector field.*

Compact Ricci solitons are the fixed points of the Ricci flow: $\frac{\partial g(t)}{\partial t} = -2Ric\,(g(t))$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Further, Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions (see e.g. [80]).

Since the early 1980s, the geometry of Ricci solitons has been the focus of attention of many mathematicians. Especially, it has become more important after Grigori Perelman applied Ricci flows to solve the long standing Poincaré conjecture proposed in 1904.

Next, we focus on the problem:

"When does a Ricci soliton on a Euclidean submanifold have the canonical vector field as its potential field?"

Such solitons have been studied in [87,92,95,96,97,98], among others. In particular, the following result classified Ricci solitons on a hypersurface M of $\mathbb{E}^{n+1}$ with its canonical vector field as the soliton field.

Theorem 8.8. [98] *Let (M^n, g, η, λ) be a Ricci soliton on a hypersurface of $M^n \subset \mathbb{E}^{n+1}$ such that the potential field η is the canonical vector field. Then M^n is one of the following five types of hypersurfaces:*

(1) A hyperplane through the origin.

(2) A hypersphere centered at the origin.

(3) An open part of a flat hypersurface generated by lines through the origin.

(4) An open part of a circular hypercylinder $S^1(r) \times \mathbb{E}^{n-1}, r > 0$.

(5) An open part of a spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}, 2 \leq k \leq n-1$.

For further results in this direction, see [87,92,95,96,97].

8.3. Yamabe Solitons With Canonical Vector Fields as Soliton Fields

Yamabe flow was also introduced by R. Hamilton [99] at the same time as the Ricci flow. It deforms a given manifold by evolving its metric according to:

$$\frac{\partial}{\partial t} g(t) = -\rho(t) g(t) \tag{8.5}$$

where $\rho(t)$ denotes the scalar curvature of the metric $g(t)$. Yamabe solitons correspond to self-similar solutions of the Yamabe flow. In dim $M = 2$, the Yamabe and Ricci flows are the same. When dim $M = n > 2$, the Yamabe and Ricci flows do not agree, since a Yamabe flow preserves the conformal class of the metric, but the Ricci flow does not in general.

A Riemannian manifold (M, g) is a *Yamabe soliton* if it admits a vector field X such that:

$$\frac{1}{2} \mathcal{L}_X g = (\rho - \lambda) g \tag{8.6}$$

where X is a vector field X and λ is a real number. Moreover, the vector field X is called a *soliton field*.

We denote the Yamabe soliton satisfying Eq. (8.6) by (M, g, X, λ).

A Riemannian manifold (M, g) is called a *quasi-Yamabe soliton* if it admits a vector field X such that

$$\frac{1}{2} \mathcal{L}_X g = (\rho - \lambda) g + \mu X^{\#} \otimes X^{\#} \tag{8.7}$$

for some constant λ and scalar function μ, where $X^{\#}$ is the dual 1-form of X. The vector field X is also called a *soliton field* for the quasi-Yamabe soliton. We denote the quasi-Yamabe soliton satisfying Eq. (8.7) by (M, g, X, λ, μ). A Yamabe (or quasi-Yamabe) soliton is called *shrinking, steady or expanding* if it admits a soliton field for which $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.

Now, we discuss Yamabe and quasi-Yamabe solitons on Euclidean submanifolds such that the potential fields are given by their canonical vector fields. Such solitons have been studied in [31,100], among others.

From [31], we have the following result:

Theorem 8.9. *If a Euclidean submanifold M* of $\mathbb{E}^m$ *is a Yamabe soliton with the canonical vector field as its soliton field, then the canonical vector field is a conformal vector field.*

As a natural extension of self-shrinker, a Euclidean submanifold M is called a *generalized self-shrinker* if it satisfies:

$$\mathbf{x}^{\perp} = fH \tag{8.8}$$

for some scalar function f. From [31], we also have the following:

Theorem 8.10. *Let M be a generalized self-similar submanifold of the Euclidean m-space $\mathbb{E}^m$. Then the canonical vector field of M is a conformal vector field if and only if M is a pseudo-umbilical submanifold.*

For Yamabe solitons, we also have the following:

Theorem 8.11. *Let (M, g) be a Riemannian manifold. Then an isometric immersion $\phi: (M, g) \to S^{m-1} \subset \mathbb{E}^m$ of M into the hypersphere of $\mathbb{E}^m$ centered at the origin is a Yamabe soliton with the canonical vector field as its soliton field if and only if M has constant scalar curvature* [100].

This theorem implies that there exist ample examples of Yamabe solitons with the canonical vector field as the soliton fields.

From [31], we have the next two results:

Theorem 8.12. *Let M be a proper hypersurface of $\mathbb{E}^{n+1}$. If $(M, g, \mathbf{x}^T, \lambda, \mu)$ is a quasi-Yamabe soliton with $\mu \neq 0$, then M is a quasi-umbilical hypersurface with the canonical vector field as its distinguished direction. Moreover, the canonical vector field is a torse-forming vector field.*

The next result follows from Theorem 8.11 and Theorem 6 in [89]:

Theorem 8.13. *Let M be a proper hypersurface of $\mathbb{E}^{n+1}$. If $(M, g, \mathbf{x}^T, \lambda, \mu)$ is a quasi-Yamabe soliton with $\mu \neq 0$, then M is an open portion of a rotational hypersurface whose axis of rotation contains the origin.*

For further results in this direction, see [31,100].

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