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Three-Valued Gödel Logic With Constants and Involution for Application to *R*-Functions

Revaz Grigolia[1,2,*]

[1] *Department of Mathematics, Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia*
[2] *Institute of Cybernetics, Georgian Technical University, Tbilisi, Georgia*

ABSTRACT

In this article we introduce a new logic: three-valued Gödel logic with constants and involution using the possibility to represent n-variable R-functions (real functions) such that the number of branches is equal to 3^{3^n} instead of 2^{2^n} in the case of classical (2-valued) logic which increases the expressibility. This many-valued logic is offered for application in the class of R-functions partitioned in branches corresponding to some Gödel logic formulas.

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1. INTRODUCTION

An *R*-function, or Rvachev function, is a real-valued function whose sign does not change if none of the signs of its arguments change; that is, its sign is determined solely by the signs of its arguments. Interpreting positive values as true and negative values as false, an *R*-function is transformed into a "companion" Boolean function (the two functions are called friends). *R*-functions are used in computer graphics and geometric modeling in the context of implicit surfaces and function representations. They also appear in certain boundary-valued problems, and are also popular in certain artificial intelligence applications, where they are used in pattern recognition. The Rvachev's method implies an ability to represent a geometric object "implicitly" by a property $Q(x)$ where $x \in R^n$, as $\Omega = \{x : Q(x) \text{ is true}\}$.

R-function is a real-valued function of real variables having the property that their signs are completely determined by the signs of their arguments, and are independent of the magnitude of the arguments. For example, the following functions satisfy this property:

- $W_1 = xyz$
- $W_2 = x + y + \sqrt{xy + x^2 + y^2}$
- $W_3 = 2 + x^2 + y^2 + 2^2$

The main idea is to find corresponding *R*-functions $f : R^n \to R$ for some Boolean function $\varphi : \{0,1\}^n \to \{0,1\}$. Roughly speaking, the Boolean functions are usually defined using logic operations ∧ (and; minimum of the two arguments), ∨ (or; maximum of two arguments), and ¬ (negation; $1-x$) on n logic variables. The Boolean function φ in the above definition is called the *companion* function of the *R*-function *f*. Informally, a real function *f* is an *R*-function if it can change its property (sign) only when some of its arguments change the same property (sign). The notion of *R*-functions is a special case of a more general concept of *R*-mapping that is associated with qualitative *k*-partitions of arbitrary domains and multi-valued logic functions [1]. We follow this idea and have proposed multiple-valued logic, namely 3-valued Lukasiewicz logic [2].

*Corresponding author. Email: revaz.grigolia@tsu.ge
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The set of all R-functions that have the same logic companion function is called a *branch* of the set of R-functions. Since the number of distinct logic functions of n arguments is 2^{2^n}, there are also 2^{2^n} distinct branches of R-functions of n arguments.

The set of R-functions is infinite. However, for applications, it is not necessary to know all R-functions; one needs only to be able to construct R-functions that belong to a specified branch. The recipes for such constructions are implied by the general properties of R-functions that follow almost immediately from their definition. Complete proofs, as well as many additional properties, can be found in [1,3].

1. The set of R-functions is closed under composition. In other words, any function obtained by composition of R-functions is also an R-function.
2. If a continuous function $f(x_1,...,x_n)$ has zeros only on coordinate hyperplanes (i.e. $f = 0$ implies that one or more $x_j = 0, j = 1,2,...,n$), then f is an R-function.
3. The product of R-functions is an R-function. If the R-function $f(x_1,...,x_n)$ belongs to some branch, and $g(x_1,...,x_n) > 0$ is an arbitrary function, then the function fg also belongs to the same branch.
4. If f_1 and f_2 are R-functions from the same branch, then the sum f_1+f_2 is an R-function belonging to the same branch.
5. If f_φ is an R-function whose logic companion function is φ, and C is some constant, then Cf_φ is also an R-function. The logic companion function of Cf is φ if $C > 0$, or $\neg\varphi$ if $C < 0$.
6. If $f_\varphi(x_1,...,x_n)$ is an R-function whose logic companion function is $\varphi(X_1,...,X_n)$ and f can be integrated with respect to x_i, then the function $\int_0^{x_i} f(x_1,...,x_n)\,dx_i$ is an R-function whose logic companion function is $X_i \Leftrightarrow \varphi(X_1,...,X_n)$.

The above list of properties is not exhaustive, but it is enough to suggest that more complex R-functions may be constructed from simpler functions. In particular, the closure under composition leads to the notion of *sufficiently complete systems* of R-functions, i.e. collections of R-functions that can be composed in order to obtain an R-function from any branch.

Theorem 1. *Let H be some system of R-functions and G be the corresponding system of the logic companion functions. The system H is sufficiently complete if the system G is complete.* [2]

It is easy to check that the following functions are R-functions (their logic companion function in parentheses):

- $C \equiv const$ (logical 1)
- $\bar{x} \equiv -x$ (logical negation ¬)
- $x_1 \wedge_1 x_1 \equiv min(x_1, x_2)$ (logical conjunction ∧)
- $x_1 \vee_1 x_1 \equiv max(x_1, x_2)$ (logical disjunction ∨)

Theorem 1 states that an R-function from any branch can be defined using composition of just these functions. But these functions are not differentiable. For applications where differentiability is important, for example in solutions of boundary value problems, another system is needed. For this one we need suitable R-conjunction and R-disjunction. Let us consider a triangle with two sides of length x_1 and x_2. The square of the third side is determined by the law of cosines as $x_1^2 + x_2^2 - 2\alpha x_1 x_2$, where α is the cosine of the angle between the two sides. It is easy to see that the function:

$$f = x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}$$

satisfies the desired properties. Moreover, the R-function corresponding to logical disjunction is:

$$f = x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}$$

In this article we suggest a new logic: 3-valued Gödel Logic with constant and involution for application to R-functions that will be a new companion function for R-functions.

2. THREE-VALUED GÖDEL LOGIC WITH CONSTANTS AND INVOLUTION

Partition $(-\infty, +\infty)$ into three sets: $(-\infty, 0), [0], +\infty)$.

Let $G_3 = (\{-1,0,1\}, \vee, \wedge, \rightarrow, \sim, -1,0,1)$ be the algebra of type $(2,2,2,1,0,0,0)$, where $x \vee y = max(x,y), x \wedge y = min(x,y)$, $\sim$ is changing of sign and $x \rightarrow y$ is given in Table 1.

x → y	1	0	-1
1	1	0	-1
0	1	1	-1
-1	1	1	1

Table 1. The → operation.

For the R-function $x \cdot y$ we can take the logic companion $\varphi(p,q) = (p \leftrightarrow q) \wedge ((p \vee q) \vee \sim (p \wedge q))$, the logical function of which in $\{-1,0,1\}$ is given in Table 2 with the following correspondence: $+ \leftrightarrow 1, 0, - \leftrightarrow -1$.

φ(p,q)	1	0	-1
1	1	0	-1
0	0	0	0
-1	-1	0	1

Table 2. Logic companion for the R-function $x \cdot y$.

For the R-function $x \cdot y$ please refer to Table 3.

x·y	+	0	−
+	+	0	−
0	0	0	0
−	−	0	+

Table 3. The R-function $x \cdot y$.

R-function $-x$ and its Gödel companion is changing sign.

R-function $c \in (0, +\infty)$ and its Gödel companion is 1.

R-function $c \in (-\infty, 0)$ and its Gödel companion is -1.

R-function 0 and its Gödel companion is 0.

R-function $max(x,y)$ and its Gödel companion is disjunction $\vee$.

R-function $min(x,y)$ and its Gödel companion is conjunction $\wedge$.

In the study of many-valued logics, one is led to consider a finite algebra *(A, o*$_1$*, ..., o*$_n$*)* that generate by composition all functions in A^{A_m} for each $m \in Z^+$. Such algebras are called *primal*. For example, Boolean algebra ({0,1},∨,∧, −.0,1) and, more generally, Post algebras $(\{0,\ldots,n - l\}, min(x,y), x + l(mod\ n))$ are primal algebras.

For primal algebras the following theorem holds:

Theorem 2. *An algebra (A, o*$_1$*, ..., o*$_n$*) is primal iff 2-generated free algebra is isomorphic to* $A^{|A|^2}$. [4,5]

Let $\mathbf{G_3}$ be the variety generated by algebra G_3 = ({−1,0,1}, ∨, ∧, →, ∼, −1,0,1). The algebra G_3^3 is depicted in Fig. 1.

Theorem 3. *The algebra* G_3^3 *is 1-generated free algebra in the variety* $\boldsymbol{G_3}$ *with free generator g = (−1,0,1).*

Proof. Let ¬x = x → −1. Let us consider the algebra:

$$G_3 = (\{-1,0,1\}, \vee, \wedge, \rightarrow, \sim, -1,0,1)$$

and its element $g = (-1,0,1)$. Now we show that g generates the algebra G_3. It can be shown this fact if we obtain the elements $(1,-1,-1), (-1,1,-1), (-1,-1,1)$ previously having the constant elements $(-1,-1,-1)$, $(0,0,0), (1,1,1)$. $\neg g = (1,-1,-1), \neg \sim g = (-1,-1,1), \neg g \vee \neg \sim g = (-1,1,-1)$.

From here we can obtain all elements of the algebra G_3 by the lattice operations ∨ and ∧. Now we show that one-variable identity $P = Q$ is true in the variety $\mathbf{G_3}$ iff the identity is true in $\boldsymbol{G_3^3}$ for generator g. Indeed, it is obvious that if $P = Q$ is true in the variety $\mathbf{G_3}$, then it is true in the algebra G_3^3. Let us suppose that $P = Q$ is not true in the variety $\boldsymbol{G_3}$. Then it is not true in the algebra G_3 for some element $a \in G_3$. Then, we can take corresponding projection $\pi_k: G_3^3 \to G_3$. where $\pi_k(g) = a$. It means that $P = Q$ is not true in G_3^3 for the generator g. From here we conclude that G_3^3 is 1-generated free algebra in the variety $\mathbf{G_3}$ with free generator $g = (-1,0,1)$. □

In the same manner the following is proven:

Theorem 4. *The algebra* $G_3^{3^2}$ *is 2-generated free algebra in the variety* $\boldsymbol{G_3}$ *with free generators* $g_1 = (1,1,1,0,0,0,-1,-1,-1), g_2 = (1,0,-1,0,-1,1,1,0,-1)$.

From this theorem holds:

Corollary 5. *The algebra* G_3 *is primal. In other words, the operations* ∨, ∧, →, ∼, −1,0,1 *of the algebra* G_3 *generate all functions in* A^{A^m} *for each* $m \in Z^+$.

From the above mentioned we conclude:

Theorem 6. *The number of branches of n-ary R-functions is equal to* 3^{3^n}.

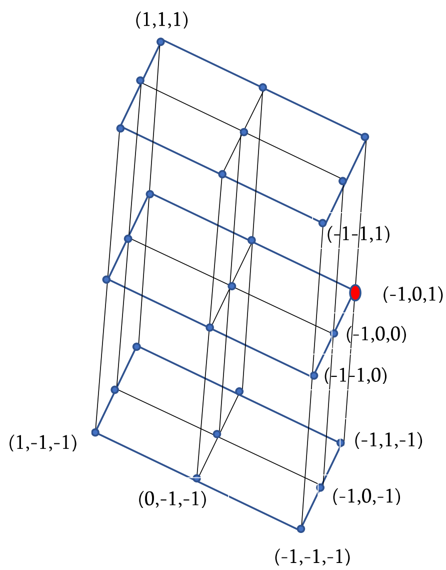

Figure 1. Graphical representation of the algebra G_3^3.

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