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PROCEEDINGS ARTICLE

Stability of Solutions in Mixed Differential Equations

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ABSTRACT

Mixed differential equations with advance and delay occur in many problems in economy, biology, physics and engineering. The concept of delay is related to the memory of systems, where past events influence current behavior. The concept of advance is related to potential future events which are known at the current time, and which could be useful for decision making. In this article various examples are given of difference and differential equations, classical, with delay, and with delay and advances (the mixed ones). It is well known that the solutions of these types of equations cannot be obtained in closed form. It is not quite clear how to formulate an initial value problem for such equations and the existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equations on the half line.

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1. DIFFERENCE AND DIFFERENTIAL EQUATIONS

1.1. Tumor Growth Cancer Model

In [1], the authors consider the following system where x' describes the density of tumor cells, y' the density of hunting predator cells and z' the density of resulting cells:

$$\begin{cases} x' = 1 + a_1 x(1-x) - k_1 xy - k_2 x \\ y' = a_2 yz - a_3 y - k_3 xy \\ z' = a_4 z(1-z) - a_5 yz - a_6 z - k_4 xz \end{cases} \tag{1}$$

- a_1 is the growth rate of tumor cells
- a_2 represents the conversion rate of the resulting cells to hunting predator cells
- a_3 is the specific loss rate of hunting predator cells
- a_4 represents the growth rate of resting cells
- a_5 is the conversion rate of resting cells to hunting predator cells
- a_6 is the specific loss rate of the resting cells
- k_1 is the rate of killing of tumor cells by hunting cells
- k_2 is the specific loss rate of tumor cells
- k_3 represents the rate of killing of hunting predator cells by tumor cells
- k_4 represents rate of killing of resting cells by tumor cells

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The equilibrium points of the system shown in Eq. (1) are:

$E_0(0,\ 0,\ 0)$

$E_1(x_1,\ 0,\ 0)$ where $x_1 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right)+\sqrt{\left(1-\frac{k_2}{a_1}\right)^2+\frac{4}{a_1}}\right] \qquad (a_1 > k_2)$

$E_2(x_2,\ 0, z_2)$ where $x_2 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right)+\sqrt{\left(1-\frac{k_2}{a_1}\right)^2+\frac{4}{a_1}}\right]$ and $z_2 = 1-\frac{a_6}{a_4}-\frac{k_4}{a_4}x_2 \quad (a_1 > k_2$ and $a_4 > a_6 + k_4x_2)$

$E_3(x_3, y_3, z_3)$ where $y_3 = \frac{1+a_1x_3(1-x_3)-k_2x_3}{k_1x_3}$ and $z_3 = \frac{a_3+k_3x_3}{a_2} \quad (a_1 > k_2)$

The equilibrium point $E_3(x_3, y_3, z_3)$ is globally asymptotically stable.

Example 1. For the system shown in Eq. (1) with:

$$a_1 = 0.6 = a_4,\ \ a_2 = 0.99,\ \ a_3 = 0.1,\ \ a_5 = 0.06,\ \ a_6 = 0.118$$
$$k_1 = 0.9,\ \ k_2 = 0.5,\ \ k_3 = 0.854,\ \ k_4 = 0.02$$

we obtain the equilibrium point E_3(*1.3213, 0.5656, 0.1186*) globally asymptotically stable (Fig. 1).

1.2. Biolarvicides Against Malaria Model

Biolarvicides are in use in several parts of the world for malaria vector control (see [2]). Consider the system:

$$\begin{cases} x' = a - \beta x m_I - dx + vy \\ y' = \beta x m_I - (v - \alpha - d)y \\ m'_S = \theta M\left(1-\frac{m_S+m_I}{L}\right) - (\theta_0 + \theta_1 B)m_S - \lambda m_S y \\ m'_I = \lambda m_I y - (\theta_0 + \theta_1 B)m_I \\ B' = \gamma B\left(1-\frac{B}{k}\right) + \gamma_1(m_S + m_I)B \end{cases} \tag{2}$$

- x represents the susceptible humans
- y represents the infected humans
- m_S represents the susceptible mosquitoes
- m_I represents the infected mosquitoes
- B represents the biolarvicide population (Fig. 2)

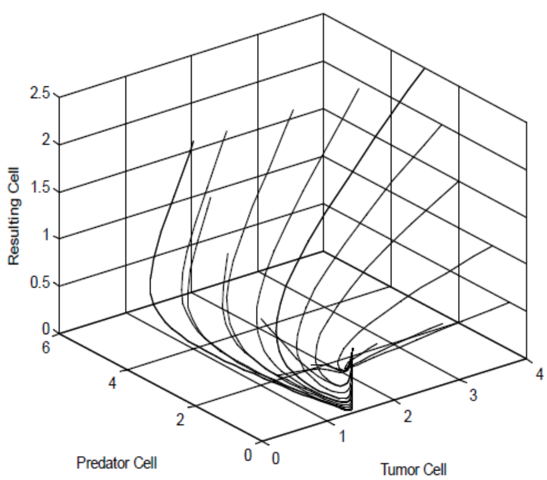

Figure 1. The equilibrium point E_3*(1.3213, 0.5656, 0.1186).*

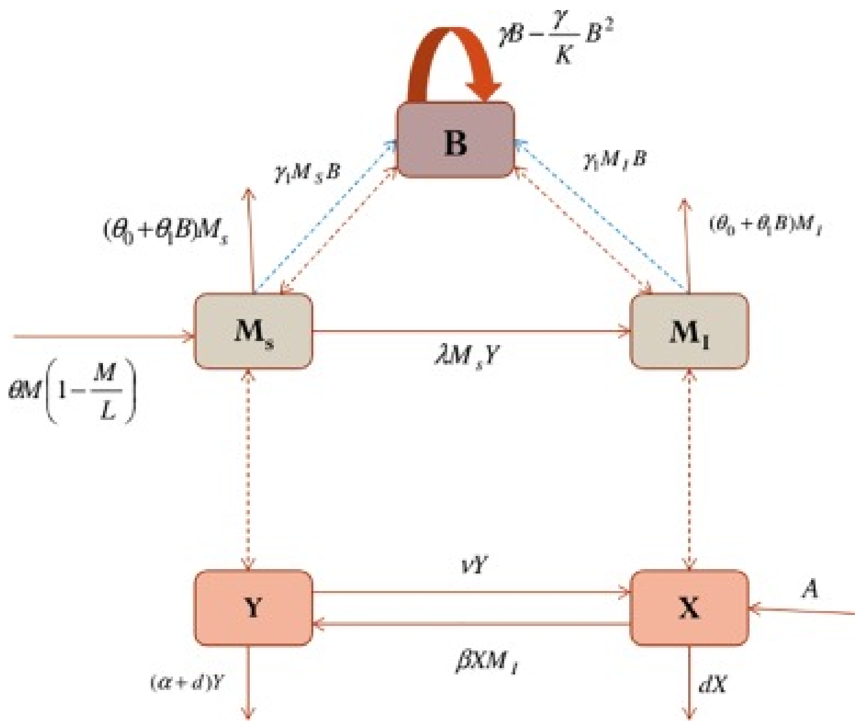

Figure 2. Schematic overview of the system. The direction of each solid line represents movement of population along that line within the same species. The bi-directional dotted lines between boxes indicate a mass-action interaction. The single directional dotted line indicates increase of bacteria population (for example, $\beta x m_I$ is a removal from x population and an addition to y population).

The equilibrium points of the system shown in Eq. (2) are:

- $E_0\left(\frac{a}{d}, 0, 0, 0, 0\right)$ Disease free, unstable
- $E_1\left(\frac{a}{d}, 0, 0, \frac{L(\theta-\theta_0)}{\theta}, 0\right)$ Disease free, unstable
- $E_2\left(\frac{a}{d}, 0, 0, 0, K\right)$ Disease free, unstable if $\frac{(\theta-\theta_0)}{\theta_1} > k$
- $E_3\,(x^*, y^*, m_S^*, m_I^*, 0)$ Endemic, unstable
- $E_4\left(\frac{a}{d}, 0, m_S^*, 0, B^*\right)$ Disease free, stable under conditions
- $E_5\,(x^*, y^*, m_S^*, m_I^*, B^*)$ Endemic, stable under conditions

2. DIFFERENCE AND DIFFERENTIAL EQUATIONS WITH DELAYS

2.1. Logistic Equations With Delays

Population density is unlikely to elicit an instant response to the per capita growth rate.

For example, the effect of food scarcity available to young immatures can only be felt later when they reach maturity expressing lower fertility rates. By designating τ the delay interval we get the delayed logistic equation:

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right)$$

or:

$$N'(t) = rN(t)\left(\frac{K - N(t-\tau)}{K + crN(t-\tau)}\right)$$

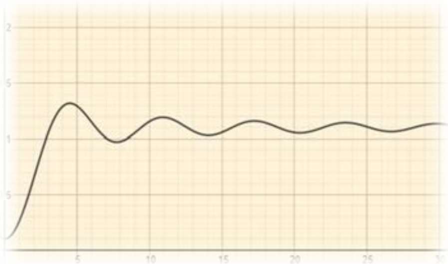 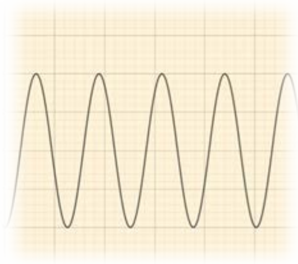

Figure 3. Oscillations.

The earliest delay model in mathematical biology is Hutchinson's equation in 1948 [3], when he modified the classical logistic equation, with a delay term to incorporate hatching and maturation periods into the model and account for oscillations, in the population of Daphnia. Such oscillations (Fig. 3) are in part due to the fact that the fertility of a parthenogenic female is determined, not merely by the population density at a given time, but also by the past densities to which it has been exposed [4].

2.2. Survival of Blood Cells

The delay differential equation:

$$N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}$$

has been used by Wazewska-Czyzewska and Lasota [5] as a model for the survival of red blood cells (Fig. 4) in an animal. Here:

- μ is the probability of death of a red blood cell
- p and γ are positive constants and are related to the production of red blood cells per unit of time
- τ is the time required to produce a red blood cell

The delay differential equation:

$$N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}$$

has a unique solution for each initial condition:

$$N(t) = \varphi(t) \qquad -\tau \leq t \leq 0$$

The positive equilibrium point is given by:

$$N^* = \frac{p}{\mu} e^{-\gamma N^*}$$

The solutions oscillate about N^* if and only if the equation $\lambda + \mu + \mu\gamma N^* e^{-\lambda\tau} = 0$ has no real roots.

Figure 4. Red blood cells.

2.3. Infinite Impulse Response Filter

A filter is a system that functions to extract the data from noise in a signal:

The defining equation for an IIR filter is the difference equation:

$$y(n)=\sum_{i=1}^{n_1} a_i y(n-i)+\sum_{i=0}^{n_2} b_i x(n-i)$$

where:

- $x(n)$ is the input signal
- $y(n)$ is the output signal
- $a_i (1 \le i \le n_1)$ and $b_i (0 \le i \le n_2)$ are real constants

In [6] it is shown that this equation is equivalent to the difference equation:

$$\Delta\left[y(n)+\sum_{l=0}^{L} c_l y(n-\rho_l)\right]+\sum_{l=0}^{n_1+max_{1\le l\le L}\,\rho_l} p(j)y(n-j)=\sum_{j=0}^{n_2+max_{1\le l\le L}\,\rho_l} q(j)x(n-j)$$

3. DIFFERENCE AND DIFFERENTIAL EQUATIONS WITH DELAYS AND ADVANCES

Mixed differential equations have mixed arguments, with delay and advance (Fig. 5). They occur in many problems in economy, biology, physics and engineering. However, this class of equations has been much less studied than other classes of functional differential equations.

Why are these types of equations a challenge? It is well known that the solutions of these types of equations cannot be obtained in closed form [7]. It is not clear how to formulate an initial value problem for such equations and the existence and uniqueness of solutions becomes complicated [8]. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equations on the half line.

Example 2. [9] Let the initial value problem with $t_0 = 0$:

$$x'(t)-x\big(h(t)\big)=0,\ \ t\ge 0,\ \ \ x(0)=0$$

With $x'(t)$ advanced on $[0,\ 2)$ and delayed on $(2, \infty)$ and with:

$$h(t)=\begin{cases} 1 & 0\le t<1 \\ -t^2+4t-2 & 1\le t<2 \\ t-|\sin(t-2)| & t\ge 2 \end{cases}$$

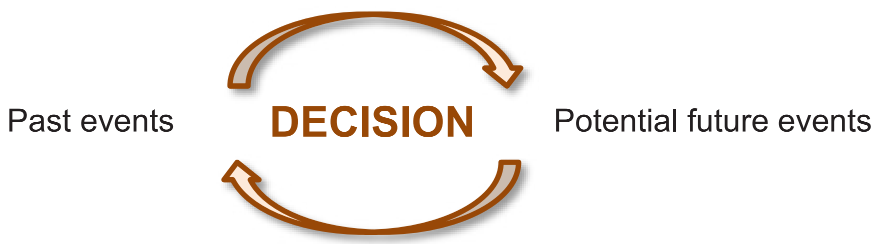

Figure 5. Mixed type equations allow to establish conditions between past and potential future events to get a good decision.

Then:

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ \dfrac{1}{2-t} & 1 \leq t < 2 \end{cases}$$

is a solution of this initial value problem on $[0,\ 2)$ which is unbounded on $[0,\ 2)$ and cannot be extended to $[2, \infty)$.

Example 3. [9] For $\alpha > 0$, the differential equation:

$$x'(t) + 2\alpha x(t) - \alpha\, x(t+1) = 0,\ \ t \geq 0,\ x(0) = 1$$

has both infinitely growing and decaying solutions $e^{\lambda t}$ on $[0, \infty)$, with λ positive and negative respectively. Indeed, for $\alpha = 0.25$:

$$x(t) = e^{-0.31812t}$$

$$x(t) = e^{2.4773t}$$

Remark: Note that for the delayed argument $h(t) \leq t$ and $0 \leq \alpha < 1$, any solution of the equation $x'(t) + 2\alpha x(t) - \alpha\, x\big(h(t)\big) = 0$ tends to zero as $t \to +\infty$.

An example with nerve conduction was studied in [10].

The equation:

$$RC\, v'(t)\ =\ F(v(t))\ +\ v(t-\tau) + v(t+t)$$

where $t \in \mathbb{R}$, $v(-\infty) = 0$ and $v(+\infty) = 1$, represents a model conduction in a myelinated nerve axon in which the myelin completely insulates the membrane, so that the potential change jumps from node to node (Fig. 6).

In the equation:

$$RCv'(t) = F(v(t)) + v(t-\tau)\ + v(t+t)$$

$v(t)$ represents the transmembrane potential at a node and the internodal delay τ represents the reciprocal of the speed of the potential wave as it propagates down the axon.

The constant r is unknown a priori and must be found simultaneously with $v(t)$. The constants R and C represent axoplasmic nodal resistivity and nodal capacity, respectively. F includes the model current-voltage relation.

Using the Ohm Law and the Taylor expansion around 0 the equation:

$$RCv'(t) = F(v(t)) + v(t-\tau)\ + v(t+t)$$

will be transformed at:

$$v'(t) = a_1 v(t) + a_2 v^2(t) + a_3 v^3(t) + v(t-\tau) - 2v(t) + v(t+t) + O(v^4)$$

Using numerical methods, we obtain the solutions in Fig. 7.

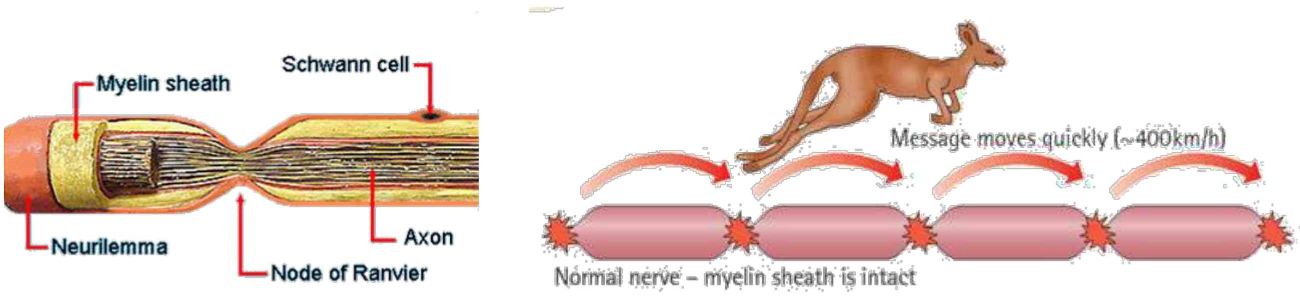

Figure 6. Nerve, consisting of axon, myelin sheath and nodes of Ranvier.

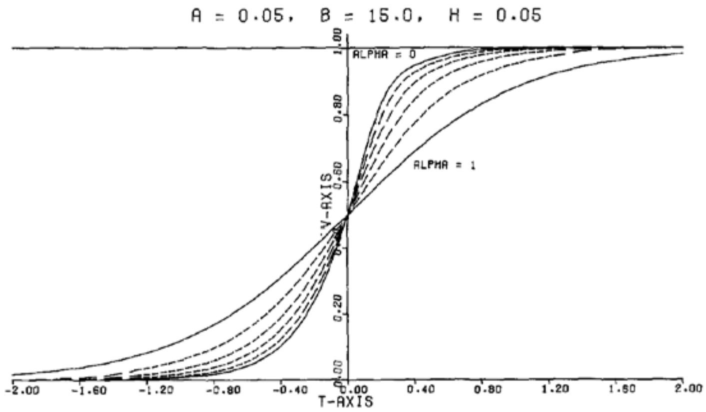

Figure 7. Graph of solutions changes when approaching target problem from test problem.

This means the rise time of the membrane potential is faster for lower threshold potential.

Example 4. The linear autonomous mixed type differential equation:

$$x'(t) = \sum_{i=1}^{p} a_i x(t - r_i) + \sum_{j=1}^{q} b_j x(t + \tau_j)$$

where a_i and b_j are non-zero real numbers and r_i and τ_j are positive real numbers, can arise in the study of traveling waves in regions with non-local interactions initiated in [11,12].

Example 5. Stability and determinacy conditions for linear mixed type functional differential equations were studied in [13]:

$$x'(t) = \int_{-a}^{b} x(t + \theta) d\mu(\theta)$$

where $\mu(\theta)$ is a real-valued function of bounded variation on $[-a, b]$.

In this study, the necessary conditions for the existence, uniqueness and stability of a solution to mixed type functional equations were obtained.

4. STABILITY AND SOLUTIONS IN DIFFERENTIAL EQUATIONS WITH DELAYS AND ADVANCES

Consider the differential equation of mixed type:

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big) d\eta(\theta)$$

where:

$$x(t) \in \mathbb{R}$$

- $r_1(\theta)$ and $r_2(\theta)$ are real non-negative continuous functions on $[-1,0]$
- $v(\theta)$ and $\eta(\theta)$ are real-valued functions of bounded variation on $[-1,0]$

We define:

$$\|r_1\| = \max\{r_1(\theta): \ -1 \le \theta \le 0\}$$

$$\|r_2\| = \max\{r_2(\theta): \ -1 \le \theta \le 0\}$$

We specify an **initial condition** of the form:

$$x(t) = \phi(t) \qquad -\|r_1\| \le t \le \|r_2\|$$

where the initial function ϕ is a given continuous real-valued function on the interval:

$$[-\|r_1\|, \|r_2\|]$$

satisfying the "**consistency condition**":

$$\phi'(0) = \int_{-1}^{0} \phi(-r_1(\theta))dv(\theta) + \int_{-1}^{0} \phi(r_2(\theta))d\eta(\theta)$$

By a **solution** of:

$$x'(t) = \int_{-1}^{0} x(t - r_1(\theta))dv(\theta) + \int_{-1}^{0} x(t + r_2(\theta))d\eta(\theta)$$

we mean a continuous function $x : [-\|r_1\|, +\infty) \to \mathbb{R}$, which is differentiable on $[0, +\infty)$ and satisfies the equation for every $t \ge 0$.

If a solution of:

$$x'(t) = \int_{-1}^{0} x(t - r_1(\theta))dv(\theta) + \int_{-1}^{0} x(t + r_2(\theta))d\eta(\theta)$$

is searched in the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$, the **characteristic equation** will be:

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta)$$

The solution of:

$$x'(t) = \int_{-1}^{0} x(t - r_1(\theta))dv(\theta) + \int_{-1}^{0} x(t + r_2(\theta))d\eta(\theta)$$

is said to be **stable** if for every $\varepsilon > 0$ there exists a number $\ell = \ell(\varepsilon) > 0$ such that, for any initial function ϕ with $\|\phi\| = \max_{-\|r_1\| \le t \le \|r_2\|} |\phi(t)| < \ell$, the solution satisfies $|x(t)| < \varepsilon$ for all $t \in [-\|r_1\|, \infty)$.

Otherwise, the solution is said to be **unstable**.

The solution is called **asymptotically stable** if it is stable in the above sense and in addition there exists a number $\ell_0 > 0$ such that, for any initial function ϕ with $\|\phi\| < \ell_0$, the solution satisfies $\lim_{t\to\infty} x(t) = 0$.

5. ESTIMATION OF SOLUTIONS AND STABILITY CRITERIA

Theorem 1. [14] Let λ_0 be a real root of the characteristic equation and:

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} \, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} \, dV(\eta)(\theta) < 1$$

$$\beta(\lambda_0) = \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} d\eta(\theta)$$

Then the solution x of:

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big) d\eta(\theta)$$

satisfies:

$$|x(t)| \leq \left[\frac{\big(1 + \mu(\lambda_0)\big)^2}{1 + \beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0; \phi) e^{\lambda_0 t}$$

with:

$$N(\lambda_0; \phi) = \max_{-\|r_1\| \leq t \leq \|r_2\|} \left| e^{-\lambda_0 t} \phi(t) \right|$$

Moreover, the solution is:

- stable if $\lambda_0 = 0$
- asymptotically stable if $\lambda_0 < 0$
- unstable if $\lambda_0 > 0$

An important Lemma is the following.

Lemma 1. [14] Assume that:

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r} \quad ; \qquad \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

and:

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1$$

where $r = max\{\|r_1\|, \|r_2\|\}$.

Then, in the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$ the characteristic equation:

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta)$$

has a unique root λ_0 and this root satisfies the property:

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)}\, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)}\, dV(\eta)(\theta) < 1$$

Corollary 1. [14] Assume that:

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r} \quad , \qquad \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

and:

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1$$

where $r = max\{\|r_1\|, \|r_2\|\}$.

Then the solution of:

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big)dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big)d\eta(\theta)$$

is:

- asymptotically stable if $v(-1) + \eta(-1) > v(0) + \eta(0)$
- unstable if $v(-1) + \eta(-1) < v(0) + \eta(0)$

Example 6. [14] Consider the equation:

$$x'(t) = \int_{-1}^{0} x\left(t - \left(\frac{\theta+1}{2}\right)\right) d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} x\left(t + \left(-\frac{\theta}{4}\right)\right) d\eta\left(-\frac{3}{2}\theta\right)$$

Notice that in this case we have:

$$r_1(\theta) = \tfrac{\theta+1}{2}; \quad v(\theta) = \tfrac{(\theta+1)^2}{4}; \quad r_2(\theta) = -\tfrac{\theta}{4}; \quad \eta(\theta) = -\tfrac{3}{2}\theta$$

The characteristic equation is:

$$\lambda = \int_{-1}^{0} e^{-\lambda\left(\frac{\theta+1}{2}\right)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{\lambda\left(-\frac{\theta}{4}\right)} d\left(-\frac{3}{2}\theta\right)$$

$$= \frac{1}{2}\int_{-1}^{0}\left[e^{-\lambda\left(\frac{\theta+1}{2}\right)}(\theta+1) - 3e^{-\frac{\lambda\theta}{4}}\right] d\theta = \frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1 - e^{-\frac{\lambda}{2}}\right) - e^{-\frac{\lambda}{2}} - 6\left(e^{\frac{\lambda}{4}} - 1\right)\right]$$

So, $F_2(\lambda) = \lambda - \frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1 - e^{-\frac{\lambda}{2}}\right) - e^{-\frac{\lambda}{2}} - 6\left(e^{\frac{\lambda}{4}} - 1\right)\right]$ (Fig. 8).

The only one root of F_2 is $\lambda \cong -0.98$.

Then, for $\lambda_0 = -0.98$ the condition of Theorem 1 is satisfied. In fact, since v is increasing on $[-1,0]$ and η is decreasing on $[-1,0]$:

$$\mu(\lambda_0) = \mu(-0.98)$$

$$\leq \left[\max_{-1\leq\theta\leq 0}\left(e^{0.98\left(\frac{\theta+1}{2}\right)}\left(\frac{\theta+1}{2}\right)\right)\right] V\left(\frac{(\theta+1)^2}{4}\right)(-1,0) + \left[\max_{-1\leq\theta\leq 0}\left(e^{0.98\left(\frac{\theta}{4}\right)}\left(-\frac{\theta}{4}\right)\right)\right] V\left(-\frac{3}{2}\theta\right)(-1,0)$$

$$= \frac{e^{\frac{0.98}{2}}}{2}\cdot\frac{1}{4} + \frac{e^{-\frac{0.98}{4}}}{4}\cdot\frac{3}{2} \cong 0.5 < 1$$

So, the solution is asymptotically stable.

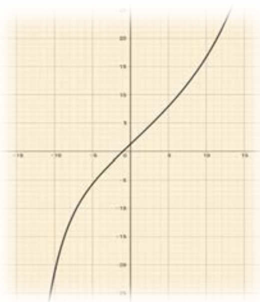

Figure 8. The function $F_2(\lambda) = \lambda - \frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1 - e^{-\frac{\lambda}{2}}\right) - e^{-\frac{\lambda}{2}} - 6\left(e^{\frac{\lambda}{4}} - 1\right)\right]$.

In this example, stability analysis can be performed using Corollary 1 of Lemma 1 without using the characteristic equation. Indeed, we get:

$$\int_{-1}^{0} e^{(\theta+1)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{\frac{\theta}{2}} d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{\theta+1} - 3e^{\frac{\theta}{2}}\right] d\theta \cong -0.68 > -2$$

$$\int_{-1}^{0} e^{-(\theta+1)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{-\frac{\theta}{2}} d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{-(\theta+1)} - 3e^{-\frac{\theta}{2}}\right] d\theta \cong -1.68 < 2$$

$$\int_{-1}^{0}\left(\frac{\theta+1}{2}\right) e^{(\theta+1)} dV\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0}\left(-\frac{\theta}{4}\right) e^{-\frac{\theta}{2}} dV\left(-\frac{3}{2}\theta\right) \leq \frac{e}{2}\cdot\frac{1}{4} + \frac{\sqrt{e}}{4}\cdot\frac{3}{2} \cong 0.96 \leq 1$$

Thus, according to Lemma 1, it states that a real root must pass in the interval $(-2\,,\ 2)$.

Finally, from Corollary 1 we obtain:

$$v(-1) + \eta(-1) = 0 + \frac{3}{2} > v(0) + \eta(0) = \frac{1}{4} + 0$$

and thus the solution is asymptotically stable.

Example 7. Consider the equation:

$$x'(t) = \int_{-1}^{0} x\left(t - \left(-\frac{\theta}{2}\right)\right) d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} x\left(t + \left(-\frac{\theta}{2}\right)\right) d\eta\left(-\frac{\theta}{4}\right)$$

Here $r_1(\theta) = -\frac{\theta}{2}$, $v(\theta) = -\frac{\theta}{4}$, $r_2(\theta) = -\frac{\theta}{2}$, $\eta(\theta) = -\frac{\theta}{4}$ and the characteristic equation is given by:

$$\begin{aligned}\lambda &= \int_{-1}^{0} e^{-\lambda\left(-\frac{\theta}{2}\right)} d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} e^{\lambda\left(-\frac{\theta}{2}\right)} d\left(-\frac{\theta}{4}\right) \\ &= -\frac{1}{4}\int_{-1}^{0}\left(e^{\frac{\lambda\theta}{2}} + e^{-\frac{\lambda\theta}{2}}\right) d\theta \\ &= -\frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right)\end{aligned}$$

So, $F_3(\lambda) = \lambda + \frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right)$.

The graph of the function F_3 (Fig. 9) shows that F_3 has two roots: $\lambda \cong -0.5$ and $\lambda \cong -11$.

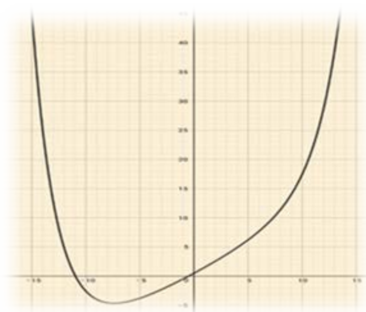

Figure 9. The function $F_3(\lambda) = \lambda + \frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right)$.

Let $\lambda = -11$:

$$|\beta(-11)| = \left| \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) - \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) \right| \cong 2.46 \leq \mu(-11)$$

So for $\lambda_0 = -11$ Theorem 1 cannot be applied.

Let $\lambda = -0.5$:

$$\mu(\lambda_0) = \mu\left(-\frac{1}{2}\right) = \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right) \leq \frac{e^{\frac{1}{4}}}{2} \cdot \frac{1}{4} + \frac{e^{-\frac{1}{4}}}{2} \cdot \frac{1}{4} \cong 0.26 < 1$$

Then for $\lambda_0 = -0.5$ the conditions of Theorem 1 are satisfied. So the solution is asymptotically stable.

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