Preface

The 1st International Symposium on Square Bamboos and the Geometree (ISSBG 2022) was held online in a videoconference format on 21-22 November 2022. The symposium was jointly organized by Genicap Beheer B.V. and Geniaal B.V., with financial support for the proceedings provided by Geniaal B.V. The DNA of both these companies is to combine pioneering research in geometry, the physical sciences and technology, through transcending boundaries and open collaborations.

ISSBG 2022 brought together botanists, biologists, technologists, physicists, applied mathematicians and geometers. Contributions spanned a wide range of subjects, but the common themes were:

- Cutting-edge research on form and shape in geometry and in the natural sciences
- Transcending boundaries

The name of the symposium symbolizes the close connection between geometry and the natural sciences.

Geometree comes from a wonderful book – *The Poetry of the Universe* – by Robert Osserman: "*We may picture the product of three thousand years of geometric inventiveness in the form of a tree – the "Geometree" – whose roots go back even further and whose branches represent the outcome of centuries of discovery and creation. With or without application, the branches and fruits of this tree are worth contemplating as a remarkable product of human imagination. The Geometree is healthy, vigorous and in full foliage, older than any redwood, and fully as majestic.*"

Square bamboos opened the door to a uniform and unified mathematical description of natural shapes. In 1993, superellipses were first used to describe the culm cross-sections of square bamboos (*Chimonobambusa* species) and other square shapes in botany. Starfish and many other natural shapes with different symmetries followed soon, through the generalization of Lamé curves to Gielis Transformations (also known as the "Superformula"), as a uniform description of natural shapes and phenomena.

The first ISSBG symposium was on invitation from our own network with contributions proposed by the participants. ISSBG 2022 was divided into three sessions: (1) Geometry; (2) Mathematics; (3) Applications in Biology and Technology. Contributions in Geometry deal with position vectors in submanifold theory, the construction of equilibrium surfaces with symmetry for anisotropic energy functions, Generalized Möbius-Listing bodies and geometric algebra using R-functions. Contributions in Mathematics deal with using nested analytic functions to compute Laplace Transforms, the stability of solutions in mixed differential equations and umbral calculus. Applications in technology involve computational optimization of antennas, applications of the superformula in CAD/CAM and technology, modeling of animal bones using superellipses and the connection to complexity theory.

Geometry

Bang-Yen Chen focuses on the role of the position vector in the study of submanifolds in Euclidean spaces, since the position vector is the most natural geometric object. They have applications everywhere in mathematics, engineering and science. Six research topics are presented in which the position vector plays an important role. It also discusses the connection between position vectors and mechanics, dynamics and D'Arcy Thompson's law of natural growth in biology.

Bennett Palmer discusses the construction of equilibrium surfaces with symmetry for anisotropic energy functions. Delaunay surfaces with constant mean curvature H (catenoids and planes for $H = 0$, unduloids, nodoids and spheres for $H \neq 0$) are based on isotropic energy functionals. If anisotropic energy functionals are considered, then the corresponding constant anisotropic surfaces with mean curvature can take many forms. The notion of Wulff shapes provides a direct link to crystallography.

Generalized Möbius-Listing surfaces and bodies generalize Möbius bands. Analogous to the cutting of the original Möbius strip, the results for this class of surfaces and bodies are completely predictable, depending on the cross-section, the number of twists and the type of cutting. In general, cutting results in interconnected and intertwined different surfaces or solids, leading to very complex systems. For an overview, see the article *A Note on Generalized Möbius-Listing Bodies*. In the article *Geometric Figures Which Appear After VV Cutting in the Radial Cross Section of Generalized Möbius-Listing Bodies*, Ilia Tavkhelidze presents new results on the number and geometric shapes appearing after cutting, in this case cutting vertex to vertex in regular polygons. This work is another step towards solving the question if it is possible to recover the GML body uniquely, given the information about the traces left on the radial cross-section.

The solution for cutting three-dimensional Generalized Möbius-Listing bodies was found using a methodology for cutting planar polygons. The resulting m-gons with the same color and shape form a single body after cutting. One way to connect the "unconnected" parts in the plane are *R*-functions, named after the Ukrainian mathematician Rvachev. Such *R*-functions can connect geometric areas separated by an infinitely small boundary based on a single partition. But it is also possible to divide the real line into an arbitrary number of partitions, leading to n-valued logics. The article by Revaz Grigolia discusses the introduction of a new three-valued Gödel logic with constants and involution for application to *R*-functions.

Mathematics

Paolo Emilio Ricci et al. report a method to compute the Laplace Transform (LT) of nested analytic functions using Bell's polynomials. Bell's polynomials have been used in many different fields, ranging from number theory to operator theory. A code for approximating the LT of general analytic composed functions is created, plus a table containing the first few values of the complete Bell's polynomials, which are then used to evaluate the LT of composed exponential functions.

Sandra Pinelas discusses the *Stability of Solutions in Mixed Differential Equations* with advance and delay, which occur in many problems in economy, biology, physics and engineering. The concept of delay is related to the memory of systems, where past events influence current behavior. The concept of advance is related to potential future events, which are known at the current time, and which could be useful for decision making. In this contribution, various examples are given of difference and differential equations, classical, with delay, and with delay and advances (the mixed ones).

Two contributions focus on umbral calculus providing new tools, embedding umbral, symbolic and operational methods. In his *A Note on Low Factorial Based Polynomials,* Giuseppe Dattoli shows that the theory of special functions and polynomials is greatly simplified by the use of algebraic methods of umbral nature. In this article, the umbral and monomiality formalism are embedded to study special polynomials expressed in terms of lower factorial polynomials.

Silvia Licciardi provides examples of umbral calculus in pure and applied mathematics, in the research of analytical or numerical solutions in different fields of mathematics, and to derive new trigonometries. The umbral image, the key element to establish the rules to replace higher transcendental functions in terms of elementary functions and to rewrite complex problems into simplified exercises, is the starting point to fix the criteria to take advantage from such a replacement, based on the Laplace and Borel transform theory. Similarly, special functions can be considered via their appropriate umbral images, for example Laguerre, Jacobi, Legendre, Tricomi-Bessel and Chebyshev or the Voigt transform.

Superellipses and Superformula in Science, Biology and Technology

Following the use of Lamé curves to model square bamboos, the Superformula generalized superellipses and supercircles for any symmetry. The superformula has found numerous applications in antenna technology, both for the optimization of antenna shapes and in computational optimization. In the article *Advanced Particle Swarm Optimization Methods for Electromagnetics* by Luciano Mescia et al., bioinspired optimization methods are used to tackle electromagnetic design problems involving the optimization of multiple parameters that are nonlinearly related to objective functions. The utilization of a novel quantum PSO-based algorithm in advanced scenarios, such as reconfigurable and shaped lens antenna synthesis, is illustrated. The hybrid modeling approach, based on the unified geometrical description enabled by Gielis Transformations, is applied in combination with a suitable quantum PSO-based algorithm, to identify the geometrical parameters for optimal antenna performance.

Bert Beirinckx was closely involved in the development of the Superformula. In *From Superellipses to Superformula and Technology* an overview is given of the various developments that led to this generalization and its extensions to 3D. The driving forces, from the earliest steps onwards, have been optimization in nature and technology. In

technology, the reduction of stresses and minimizing material use, optimizations of gear technology and engines, and applications in computer graphics and CAD/CAM are discussed, with a view towards the future. In biology, stress reduction in plant roots is discussed. Modeling roots as superellipses unveils the role of the quiescent center in roots. His work also led to the development of various formulae useful in engineering and the study of natural shapes.

Luděk Spíchal discusses the use of ellipses and superellipses to model the bones of vertebrates, where some have slender marrow cavities and relatively thick walls, while others are built oppositely. This study proposes new models based on the ellipse and superellipse, which more closely approximate the actual shape of the cross-sections. The derived models are compared with a small sample of the cross-sections of a wild boar's limb bones (humerus, femur).

Superellipses and the Superformula have been tested on more than 40,000 biological specimens, including tree rings, plant leaves, stomata of plants, avian eggs, starfish and fruits of *Ginkgo* and *Koelreuteria*. Due to the work of Peijian Shi at Nanjing Forestry University, the Superformula and superellipses have become a useful methodology to study shapes and forms in nature. All methods have been published as open-source software [1,2]. In these developments, Karl Niklas of Cornell University played a pivotal role. He was also the Editor of the *American Journal of Botany* which published the original paper describing the Superformula [3]. The use of the Superformula and superellipses to model diatoms has been the subject of a talk at ISSBG 2022, by Edoardo De Tommasi and Alessandra Rogato. Their article *The Diatom Frustule: Morphogenesis and Role in Light Manipulation* has been published in the journal *Growth and Form* [4].

In *Conquering Mount Improbable,* the monomials x^n, y^n of Lamé curves are viewed as the outermost entries of Pascal's Triangle and are juxtaposed to the common methods in the sciences to consider the Law of Large Numbers. But both can be seen as dual methods. In this context, it is to be expected that the notions of information, complexity, simplicity and redundancy benefit from this different viewpoint. Furthermore, when the monomials are considered as ratios, superparabolas and power laws can be framed in the same framework. Examples are given in biology, economy and ecology, and connections to geometry are discussed.

We wish to convey our gratitude to all the participants and sincerely hope that this fine collection can inspire many researchers. Finally, many thanks to Remco de Boer, the founder of Athena Publishing, and to the Athena editorial team for their support in the publication of these proceedings.

Johan Gielis & Simone Brasili

References

[1] P. Shi, J. Gielis, B.K. Quinn, K.J. Niklas, D.A. Ratkowsky, J. Schrader, H. Ruan, L. Wang, Ü. Niinemets. 'Biogeom': An R Package for Simulating and Fitting Natural Shapes. Annals of the New York Academy of Sciences, 2022, 1516(1): 123–134. https://doi.org/10.1111/nyas.14862

[2] P. Shi, J. Gielis, B.K. Quinn. Biogeom: Biological Geometries. The Comprehensive R Archive Network (CRAN), CRAN package, version 1.3.7, published 21 October 2023. https://CRAN.R-project.org/package=biogeom

[3] J. Gielis. A Generic Geometric Transformation That Unifies a Wide Range of Natural and Abstract Shapes. American Journal of Botany, 2003, 90(3): 333–338. https://doi.org/10.3732/ajb.90.3.333

[4] E. De Tommasi, A. Rogato. The Diatom Frustule: Morphogenesis and Role in Light Manipulation. Growth and Form, 2023, 4(1-2): In Press, Corrected Proof, 8pp. https://doi.org/10.55060/j.gandf.230912.001