Preface

The Pythagorean theorem being linked to the concept of orthogonality has always been of fundamental importance in applied mathematics, since its origins, when the Pythagorean numerical triples were used to determine the right angle necessary for the construction of temples, altars and buildings. The theorem was first extended to the case of Euclidean spaces with n dimensions and subsequently, with the advent of functional analysis [86], to Hilbert spaces through the so-called Parseval equality.

After a concise introduction of the analysis concepts necessary for understanding the text and a brief biography of Jean-Baptiste Joseph Fourier, we recall the classical concepts of convergence for the Fourier series [100] and their validity for functions satisfying suitable conditions. It was only in 1966 that L. Carleson was able to prove the theorem about the almost everywhere convergence of the Fourier series and for his proof he was awarded the Abel prize in 2006.

We recall that in 1926 Andrey Kolmogorov had shown that there exist continuous functions for which the corresponding Fourier series fails to converge anywhere and so is numerically meaningless. Hence there were strong doubts about the possibility of proving the theorem proclaimed by Fourier, which had been so disputed by the great mathematicians of his time. Furthermore, it is recalled that the convergence (in quadratic mean) of the series of a given function occurs towards the function itself if the chosen orthonormal basis constitutes a complete system, or towards the orthogonal projection of the function on the linear manifold generated by the basis functions in the opposite case.

Finally, the fact is highlighted that the Pythagorean equation, in its extension due to G. Lamé, inspired one of us to introduce the so-called superformula or generalization of the Lamé formula that unifies, through the choice of appropriate parameters, the most different natural and abstract forms. This generalization became known as Gielis Transformations. In geometry they lead naturally to the framework of Riemann-Finsler geometry (which could be called more aptly Riemann-Finsler-Minkowski-Lamé geometry). In biology several tests of over 40,000 samples have provided the verification that it is an excellent model to study natural shapes as diverse a starfish, tree rings, seeds, leaves and cross sections of plant stems and petioles.

This suggests that these curves, rather than circles and straight lines, should be a preferred model for the geometrization of nature. A main question then is whether a new type of calculus is needed. But that does not seem to be the case. After all, the transformations make use of the classical trigonometric functions and the four main operations in mathematics. Due to the work of the first author, it was shown that the classic Fourier projection method can be used to study boundary value problems on Gielis domains and more generally on normal polar domains. This is not restricted to $2D$ domains, but works also for $3D$ domains and up. In $3D$ it generalizes the classic coordinate systems (spherical, cylindrical, toroidal, ...).

Moreover, shapes and phenomena can be studied within the framework of submanifold theory, whereby shapes and the environment in which they reside, live and grow, both play an active role in the genesis and evolution of shapes and phenomena. In his Origin of Species, Darwin wrote: "*There is grandeur in this view of life, ..., from so simple a beginning endless forms most beautiful and most wonderful have been, and are being, evolved*". We have the same here, with a simple beginning, but it can be put into a broader framework not only for living beings. It is in line with D'Arcy Thompson's: "*So the living and the dead, things animate and inanimate, we dwellers in the world and the world in which we dwell are bound alike by physical and mathematical law*". It is a continuation of the old Greek strive for Harmony, rather than the evolutionary struggle of survival of the fittest.

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