Chapter 2. Recalling Sequences & Series

We recall, in a schematic way, the basic definitions about sequences and series [62].

A **sequence** is a function defined on the set $\mathbf{N}$ of integers ($n \in \mathbf{N}$).

A property is said to be *definitively* true if it holds for all integer indices greater than a given value.

The **sequence** $\frac{1}{n}$ converges to zero. We write: $\frac{1}{n} \to 0$.

The **sequence** n diverges positively. We write: $n \to +\infty$.

The **sequence** $\{(-1)^n\}$ is indeterminate: it does not admit a limit.

The **sequence** $\{a_n\}$ converges to the limit $\ell \Leftrightarrow |a_n - \ell| \to 0$.

The **sequence** $\{a_n\}$ is convergent if and only if $\forall \varepsilon > 0$ it results definitively that $|a_n - a_m| < \varepsilon$ (Cauchy's convergence criterion).

The **sequence** $\{a_n\}$ diverges positively $\Leftrightarrow \forall K > 0$ it results definitively that $|a_n| > K$.

A **series** is the sum of the terms of a sequence: $\sum_{k=1}^{\infty} a_k$.

The study of **series** reduces to that of the **sequence** of partial sums: $s_n = \sum_{k=1}^{n} a_k$.

If $\{s_n\} \to s$ the **series** is convergent and its sum is s.

If $\{s_n\} \to +\infty$ the **series** diverges positively.

If $\{s_n\}$ does not have a limit the **series** is said to be indeterminate.

The **series** of Zeno's paradox is convergent: $\sum_{k=0}^{\infty} 1/2^n = 2$.

The harmonic **series** is positively divergent: $\sum_{k=1}^{\infty} 1/k = +\infty$.

The **series** $\sum_{k=0}^{\infty}(-1)^k$ is indeterminate.

If $\{a_n\}$ depends on $x \in (a,b)$ we have **sequences** or **series** of functions.

For example, the **series** of functions $\sum_{k=1}^{\infty} a_k(x)$ converges to $S(x)$ in (a,b) if $\forall x \in (a,b)$ we have:

$$|S_n(x) - S(x)| = |\sum_{k=1}^{n} a_k(x) - S(x)| \to 0$$

The convergence of this **series** is uniform in $[\alpha,\beta] \subset (a,b)$ if:

$$\max_{x\in[\alpha,\beta]} |S_n(x) - S(x)| \to 0$$

It is proven that the limit of a uniformly convergent **sequence** of continuous functions is a continuous function.

Example: consider the geometric **series** $\sum_{k=0}^{\infty} x^k$.

Recalling the equation $1-x^n = (1-x)(1+x+x^2+\cdots+x^{n-1})$, assuming $x \neq 1$, it follows that:

$$S_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

Since:

$$x^n \to \begin{cases} 0 & if\ |x| < 1 \\ +\infty & if\ x > 1 \\ \text{indeterminate} & if\ x \leq -1 \end{cases}$$

we can conclude that:

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & if\ |x| < 1 \\ +\infty & if\ x \geq 1 \\ \text{indeterminate} & if\ x \leq -1 \end{cases}$$

Moreover, the convergence cannot be uniform in $[-1,1]$, but only in intervals of the type $[\alpha,\beta] \subset (-1,1)$.