Chapter 3. Hilbert and Metric Spaces

Hereafter, for simplicity of writing, we will abandon the notation in **bold** to denote vectors.

The vector space $\mathcal{H}$ on $\mathbf{R}$ is said to be equipped with a scalar product if there is a law that each pair x, y of elements of $\mathcal{H}$ is associated with a real number, denoted by the symbol (x, y), so that the following properties hold $\forall x, y, z \in \mathcal{H}, \forall \alpha, \beta \in \mathbf{R}$:

$$\begin{array}{ll}(1) & (x,x) \geq 0 \\ (2) & (x,x) = 0 \quad \Leftrightarrow \quad x = 0 \quad \text{(zero vector)} \\ (3) & (\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z) \\ (4) & (x,y) = (y,x)\end{array}$$

By the above properties it is even possible to prove the following ones:

$$\begin{array}{ll}(5) & (\alpha x, \alpha x) = |\alpha|^2 (x,x) \\ (6) & |(x,y)| \leq \sqrt{(x,x)}\sqrt{(y,y)} \\ (7) & \sqrt{(x+y, x+y)} \leq \sqrt{(x,x)} + \sqrt{(y,y)} \qquad \forall x, y \in \mathcal{H}\end{array}$$

In (6) (the Cauchy-Schwarz inequality) the equality holds if and only if x and y are linearly dependent. In (7) the equality holds only if $x = 0$ or $x = \alpha y$ with $\alpha \geq 0$.

From the above-mentioned properties it immediately follows that every space endowed with a scalar product is a normed space (and therefore also metric) with the definition of norm given by:

$$\|x\| := \sqrt{(x,x)}$$

In particular, from property (7) it follows that every space equipped with a scalar product is a normed space. If the thus obtained normed space $\mathcal{H}$ is complete, then it is called a Hilbert space. Completeness means that every sequence that satisfies the Cauchy convergence criterion converges to a vector that belongs to the space.

Two vectors x, y of $\mathcal{H}$ are said to be orthogonal if their scalar product vanishes: $(x, y) = 0$.

3.1 Infinite Dimensional Vector Spaces; The Space L^2_w

The space constituted by the polynomial functions has an infinite dimension, since whatever n the vectors $\{1, x, x^2, x^3, \dots, x^n\}$ are linearly independent. Indeed a linear combination of them (i.e. a polynomial) is identically zero if and only if all the coefficients of the combination are zero.

The space $L^2_w(a, b)$, with $w(x)$ a non-negative real weight function not vanishing almost everywhere in (a, b), is made of almost continuous real functions in (a, b) and such that the integral of the function $f^2(x)w(x)$ in (a, b) is bounded [86].

The scalar product is defined by:

$$(f, g)_w := \int_a^b f(x)g(x)w(x)dx$$

Note the transition from discrete to continuous: when the vectors $\mathbf{u}, \mathbf{v}$ have a discrete number n of components, their scalar product with weight w is the sum of products $u_1v_1w_1 + u_2v_2w_2 + \cdots + u_nv_nw_n$.

When the functions $f(x)$, $g(x)$ and $w(x)$ are defined on the interval (a, b), their components must be interpreted as the infinite values assumed in (a, b) and the scalar product is transformed into the integral of the products $f(x)g(x)w(x)$:

$$\sum_{k=1}^{n} u_k v_k w_k \quad \to \quad \int_a^b f(x)g(x)w(x)dx$$

A set (also called system) of functions of a Hilbert space is said to be complete if it is possible to approximate any function of space to less than a predetermined ε number by means of a finite linear combination of elements of the system.

Suppose we have an orthonormal complete system of functions $\{u_n(x)\}$, $(n = 0, 1, 2, \dots)$, such that $\forall h, k$:

$$(u_h, u_k)_w = \int_a^b u_h(x) u_k(x) w(x) dx = \delta_{h,k}$$

Then, expanding a function $f(x)$ in (a, b) by means of the *uniformly convergent* series:

$$f(x) = f_0 u_0(x) + f_1 u_1(x) + \cdots + f_n u_n(x) + \cdots$$

proceeding analogously to the discrete case, we find that the components f_k of the function $f(x)$, with respect to the aforementioned basis, are given by the numbers:

$$f_k = (f, u_k)_w = \int_a^b f(x) u_k(x) w(x) dx$$

which are called the Fourier coefficients of the function $f(x)$.