Chapter 5. Applications

5.1 The Dirichlet Problem for the Laplace Equation in a Circular Domain

Consider in the plane x, y the circle C, centered at the origin O and with radius r. We want to construct a function *harmonic* regular in C, which takes on assigned values on the boundary ∂C. It is therefore necessary to solve the problem:

$$\begin{cases} \Delta_2 u \equiv \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & (x, y) \in C - \partial C \\ u = f(x, y) & \text{on } \partial C \end{cases} \tag{5.1}$$

It is usual to introduce the polar coordinates ($x = \rho \cos \varphi$, $y = \rho \sin \varphi$) and to translate the problem (5.1) into the equivalent one (with a simplified notation):

$$\begin{cases} \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \varphi^2} = 0 & \rho \in [0, r),\ \varphi \in [0, 2\pi] \\ u(r, \varphi) = f(\varphi) & \forall \varphi \in [0, 2\pi] \end{cases} \tag{5.2}$$

with the regularity condition for the solution u also for $\rho = 0$ and with $f(0) = f(2\pi)$. Assuming that there exists a solution $u(\rho, \varphi)$ of this problem, for any fixed $\rho < r$ as a function of φ it can certainly be expanded in a Fourier series. Therefore, we have:

$$u(\rho, \varphi) = \frac{1}{2}\, a_0(\rho) + \sum_{k=1}^{\infty} \left(a_k(\rho) \cos k\varphi + b_k(\rho) \sin k\varphi\right) \tag{5.3}$$

with the coefficients $a_k(\rho)$ and $b_k(\rho)$ respectively given by:

$$a_k(\rho) = \frac{1}{\pi} \int_0^{2\pi} u(\rho, \xi) \cos k\xi d\xi, \quad b_k(\rho) = \frac{1}{\pi} \int_0^{2\pi} u(\rho, \xi) \sin k\xi d\xi \tag{5.4}$$

For the determination of these coefficients there are several methods: separation of variables, transforms or computation of coefficients by using identities of trigonometric series. Here we will operate in a "formal" way which can be verified. Assuming $\forall \rho < r$ that we can derive *twice* by series, substituting (5.3) into (5.2) we get:

$$\begin{aligned}
&\frac{1}{2}\left[a_0''(\rho) + \frac{1}{\rho}\, a_0'(\rho)\right] + \sum_{k=1}^{\infty}\left[a_k''(\rho) + \frac{1}{\rho}\, a_k'(\rho) - \frac{k^2}{\rho^2}\, a_k(\rho)\right]\cos k\varphi \\
&\quad + \left[b_k''(\rho) + \frac{1}{\rho}\, b_k'(\rho) - \frac{k^2}{\rho^2}\, b_k(\rho)\right]\sin k\varphi = 0
\end{aligned} \tag{5.5}$$

a relation that must be verified identically $\forall \rho < r$, $\forall \varphi \in [0, 2\pi]$. For this to happen, the coefficients of the Fourier series (5.5) must vanish and therefore we find:

$$\begin{cases}
a_0''(\rho) + \dfrac{1}{\rho}\, a_0'(\rho) = 0 \\[2mm]
a_k''(\rho) + \dfrac{1}{\rho}\, a_k'(\rho) - \dfrac{k^2}{\rho^2}\, a_k(\rho) = 0 & (k = 1, 2, \dots) \\[2mm]
b_k''(\rho) + \dfrac{1}{\rho}\, b_k'(\rho) - \dfrac{k^2}{\rho^2}\, b_k(\rho) = 0 & (k = 1, 2, \dots)
\end{cases}$$

These are all equations of the Euler type, with the peculiarity of claiming regular solutions even for $\rho = 0$. From:

$$\begin{cases}
a_0(\rho) = A_0 + A_0^* \log \rho \\
a_k(\rho) = A_k \rho^k + A_k^* \rho^{-k} & (k = 1, 2, \dots) \\
b_k(\rho) = B_k \rho^k + B_k^* \rho^{-k} & (k = 1, 2, \dots)
\end{cases}$$

it follows that $A_0^* = A_k^* = B_k^* = 0,\ k = 1, 2, \dots$; and lastly that:

$$a_0(\rho) = A_0, \quad a_k(\rho) = A_k \rho^k, \quad b_k(\rho) = B_k \rho^k \quad (k = 1, 2, \dots)$$

From Equation (5.4) for $\rho = r$ (as a consequence of the claimed regularity) it follows that:

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} f(\xi) d\xi$$

$$A_k = a_k(r) r^{-k} = \frac{1}{\pi r^k} \int_0^{2\pi} f(\xi) \cos k\xi d\xi \quad (k = 1, 2, \dots)$$

$$B_k = b_k(r) r^{-k} = \frac{1}{\pi r^k} \int_0^{2\pi} f(\xi) \sin k\xi d\xi \quad (k = 1, 2, \dots)$$

and then:

$$a_0(\rho) = \frac{1}{\pi} \int_0^{2\pi} f(\xi) d\xi$$

$$a_k(\rho) = \frac{1}{\pi} \left(\frac{\rho}{r}\right)^k \int_0^{2\pi} f(\xi) \cos k\xi d\xi \quad (k = 1, 2, \dots)$$

$$b_k(\rho) = \frac{1}{\pi} \left(\frac{\rho}{r}\right)^k \int_0^{2\pi} f(\xi) \sin k\xi d\xi \quad (k = 1, 2, \dots)$$

We thus formally arrive at the expression of the solution:

$$\begin{aligned} u(\rho, \varphi) = & \frac{1}{2\pi} \int_0^{2\pi} f(\xi) d\xi \\ & + \frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\rho}{r}\right)^k \left[\int_0^{2\pi} f(\xi) \cos k\xi d\xi \ \cos k\varphi \right. \\ & \left. + \int_0^{2\pi} f(\xi) \sin k\xi d\xi \ \sin k\varphi \right] \\ = & \frac{1}{2\pi} \int_0^{2\pi} f(\xi) d\xi + \frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\rho}{r}\right)^k \int_0^{2\pi} f(\xi) \cos k(\varphi - \xi) d\xi \end{aligned}$$

The appropriate checks can be carried out on this expression.

5.2 The Heat Problem

Let us start with some definitions necessary for the understanding of what follows.

Gamma and Bessel functions

The Gamma function is the extension of the factorial to non-integer values of the number $n \in \mathbb{N}^+$. For $x \neq -n$ it is defined as:

$$\Gamma(x) := \int_0^{+\infty} e^{-t}\, t^{x-1} dt \qquad (x > 0) \tag{5.6}$$

In fact, we have:

$$\Gamma(1) := 1\,, \qquad \Gamma(x+1) := x\Gamma(x) \quad \Longrightarrow \quad \Gamma(n+1) = n!$$

The Bessel functions of the first kind J_n, together with those of the second kind Y_n, are widely used in the solutions of mathematical physics problems. They can be defined as solutions of the differential equation:

$$x^2 y'' + x y' + \left(x^2 - n^2\right) y = 0 \tag{5.7}$$

We get the explicit expression of J_n in the form:

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+n}}{k!\ (k+n)!} \tag{5.8}$$

which extends to the case of the real values p of the index by replacing the factorial with the Gamma function:

$$J_p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+p}}{k!\ \Gamma(k+p+1)}$$

One of the well known applications of the Bessel functions [1] is related to the separation of variables in the partial differential equation representing the heat equation for a circular plate.

In fact, denoting by B a circular domain of radius $r = 1$ centered at the origin, by ∂B the relevant boundary, by κ a constant representing the known diffusivity and by $f(x,y) \in C^0(B)$ the initial temperature, the solution:

$$u(x,y,t) \in [C^2(\overset{\circ}{B}) \times C^1(\mathbf{R}^+)] \cap C^0[\bar{B} \times \mathbf{R}^+]$$

of the differential problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) & \text{in} \quad \overset{\circ}{B} \\ u(x,y,t)|_{(x,y)\in\partial B} = 0 & u\,(x,y,0) = f\,(x,y) \end{cases} \tag{5.9}$$

putting:

$$U(\rho,\theta,t) = u(\rho\cos\theta, \rho\sin\theta, t)\,, \qquad F(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta) \tag{5.10}$$

can be represented by the Fourier expansion in terms of exponential, circular and Bessel functions:

$$\begin{aligned} u(x,y,t) &= U(\rho,\theta,t) \\ &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(A_{m,k}\cos m\theta + B_{m,k}\sin m\theta)J_m\left(j_k^{(m)}\rho\right) \\ &\quad \times \exp\left[-\left(j_k^{(m)}\right)^2 \kappa t\right] \end{aligned} \tag{5.11}$$

where the coefficients $A_{m,k}, B_{m,k}$ are given by:

$$\begin{cases} A_{0,k} = \dfrac{1}{\pi\left[J_1(j_k^{(0)})\right]^2}\displaystyle\int_0^1 \zeta\left[\int_0^{2\pi} F(\zeta,\tau)\ d\tau\right] J_0(j_k^{(0)}\,\zeta)\ d\zeta \\ A_{m,k} = \dfrac{2}{\pi\left[J_{m+1}(j_k^{(m)})\right]^2}\displaystyle\int_0^1 \zeta\left[\int_0^{2\pi} F(\zeta,\tau)\cos m\tau\ d\tau\right] J_m(j_k^{(m)}\,\zeta)\ d\zeta \\ B_{m,k} = \dfrac{2}{\pi\left[J_{m+1}(j_k^{(m)})\right]^2}\displaystyle\int_0^1 \zeta\left[\int_0^{2\pi} F(\zeta,\tau)\sin m\tau\ d\tau\right] J_m(j_k^{(m)}\,\zeta)\ d\zeta \end{cases} \tag{5.12}$$

and $j_k^{(m)}$ denote the zeros of the Bessel function J_m.

5.3 The Wave Problem

Another well known application of the Bessel functions [1] is related to the separation of variables in the partial differential equation representing the free vibrations of a circular membrane (drumhead). Denoting by B a circular domain of radius $r = 1$ centered at the origin, by ∂B the relevant boundary, by $a = \sqrt{\tau/\mu}$ a suitable constant (where τ denotes the tension and μ the density) and by $f(x,y) \in C^0(B)$ the initial displacement, the solution $u \in C^2(B - \partial B) \cap C^0(\bar{B})$ of the differential problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) & \text{in} \quad \mathring{B} \\ u(x,y,t)\big|_{(x,y)\in\partial B} = 0 & u(x,y,0) = f(x,y)\,, \quad u_t(x,y,0) = 0 \end{cases} \tag{5.13}$$

putting:

$$U(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta)\,, \qquad F(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta) \tag{5.14}$$

can be represented by the Fourier expansion in terms of Bessel functions:

$$\begin{aligned} U(\rho,\theta,t) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} & J_m\left(j_k^{(m)}\rho\right)\cos\left(j_k^{(m)}at\right) \\ & \times \left(A_{m,k}\cos m\theta + B_{m,k}\sin m\theta\right) \end{aligned} \tag{5.15}$$

where the coefficients $A_{m,k}, B_{m,k}$ are given by:

$$A_{m,k} = \begin{cases} \dfrac{2}{\pi\left[J_{m+1}(j_k^{(m)})\right]^2} \displaystyle\int_0^1 \int_0^{2\pi} \rho F(\rho,\theta) J_m\left(j_k^{(m)}\rho\right)\cos m\theta \, d\theta d\rho \\ \hfill (m = 1, 2, \dots) \\ \dfrac{1}{\pi\left[J_1(j_k^{(0)})\right]^2} \displaystyle\int_0^1 \int_0^{2\pi} \rho F(\rho,\theta) J_0\left(j_k^{(0)}\rho\right) d\theta d\rho \quad (m = 0) \end{cases}$$

$$B_{m,k} = \frac{2}{\pi \left[J_{m+1}(j_k^{(m)})\right]^2} \int_0^1 \int_0^{2\pi} \rho F(\rho,\theta) J_m \left(j_k^{(m)} \rho\right) \sin m\theta \, d\theta d\rho$$

$$(m = 1, 2, \dots) \qquad (5.16)$$

and $j_k^{(m)}$ denote the zeros of the Bessel function J_m.

Moreover, the eigenvalues of a vibrating circular membrane are related to the zeros of the Bessel functions, since the relevant elementary frequencies are given by:

$$f_{m,k} = \frac{j_k^{(m)} a}{2\pi} \qquad (m = 0, 1, \dots; \quad k = 1, 2, \dots)$$