

## PROCEEDINGS ARTICLE

Construction of Equilibria With Symmetry for  
Anisotropic Energy FunctionalsBennett Palmer<sup>1,\*</sup><sup>1</sup> Idaho State University, Pocatello, United States

## ABSTRACT

In this article we discuss the construction of equilibrium surfaces with symmetry for anisotropic energy functionals.

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## 1. INTRODUCTION

Anyone who has cut meat or sawed wood is aware that it is easier to cut or saw in some directions than in others. If we consider meat or wood as an idealized horizontal three-dimensional slab, then we observe that the amount of energy required to create a unit area of surface, depends entirely on the direction of the piece of surface.

The same phenomena is encountered when cutting through any crystalline solid, or more generally, through any material in an ordered phase. Since intermolecular bonds are stronger in certain directions than in others, more work must be done to cut along these directions and thus to create a unit area of surface along them. To account for this Josiah Gibbs introduced a notion of anisotropic surface energy by defining a density  $\gamma(\nu)$  which is the unit energy per unit area of a surface element having unit normal  $\nu$ . The total energy of the surface is then given by summing this over the surface:

$$\mathcal{F} = \int_{\Sigma} \gamma(\nu) d\Sigma \quad (1)$$

The deviation of  $\gamma$  from being a constant is a measure of the anisotropy of the surface.

According to the principles of thermodynamics, the tendency of a surface to minimize its surface energy is the most important factor in determining the surface geometry. Of course, this minimization must take into account any constraints which are imposed, for example boundary conditions, volume constraints. This tendency of the surface to minimize energy yields a differential equation, known as the *Euler-Lagrange equation*, which gives a necessary condition which any equilibrium surface must satisfy. The first one to apply the calculus of variations in relation to anisotropic surface energy was Pierre Curie [1].

At small scales, the shape of biological organisms is driven by their surface energy and not their bulk energy. In his epic work “*On Growth and Form*” [2], D’Arcy Thompson makes the case that biological organisms are, first and

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foremost, physical entities and therefore their morphology is, at least, partially driven by the laws of physics. He also makes the claim that the anisotropic nature of the constituent materials of these organisms is responsible for the deviation of their shapes from being spherical in shape.

Although he does not explicitly use an energy of the form in Eq. (1), he employs the anisotropic Young-Laplace equation:

$$\frac{T}{R} + \frac{T'}{R'} = \text{constant} \quad (2)$$

to characterize their equilibrium shapes in the absence of external forces. Eq. (1) is exactly the so-called Euler-Lagrange equation which characterizes the equilibrium states of Eq. (2) with a volume constraint. Here  $R, R'$  are the radii of curvature of the surface of the organism,  $T, T'$  are orthogonally directed stresses. (This expression may not be adequate to describe the general situation since it presupposes that the direction of the stresses in the material coincide with the principal directions of the surface.)

Remarkably, the type of energy defined in Eq. (1) possesses a canonical equilibrium known as the *Wulff shape*. We can consider the density  $\gamma$  as a function defined on the unit two-dimensional sphere  $S^2$  in three-dimensional space. As each point  $v \in S^2$ , we consider the half-space  $\{X \in \mathbf{R}^3 \mid X \cdot v \leq \gamma(v)\}$ . The intersection of all such half-spaces defines a convex body whose boundary  $W$  is called the Wulff shape, i.e.:

$$W = \partial \bigcap_{v \in S^2} \{X \cdot v \leq \gamma(v)\} \quad (3)$$

A series of results, collectively known collectively as Wulff's Theorem, state that the surface  $W$  is the unique absolute minimizer of the free energy (Eq. (3)) among all closed surfaces enclosing the same volume as  $W$  [3,4] (for other values of the volume, the minimizer is the appropriate rescaling of  $W$ ). Some biographical information on Georg Wulff can be found in [5]. In the special case  $\gamma \equiv 1$ , Wulff's Theorem is just the classical three-dimensional isoperimetric inequality which states that the round sphere is the unique minimizer of surface area for a given volume. The Ukrainian crystallographer Wulff first stated this result in the case where  $W$  and all competing surfaces are "crystals", i.e. surfaces that are piecewise linear. Over the years, Wulff's result has been generalized to the point where it is now known to hold (in any dimension) when  $W$  is an arbitrary convex surface and the comparison surfaces need only be surfaces in some measure theoretic sense.

Besides the result that the Wulff shape  $W$  is the unique minimizer, there are results characterizing  $W$  as the unique stable closed equilibrium surface, the unique embedded closed equilibrium and the unique genus zero, closed equilibrium for the surface energy (Eq. (3)).

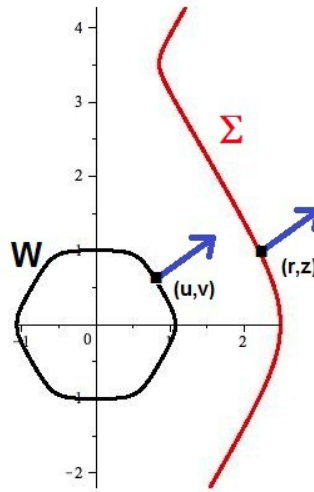
## 2. DELAUNAY SURFACES

The axially symmetric constant mean curvature surfaces were found by Charles Delaunay [6]. Delaunay found that if a conic section is rolled without slipping along a line, the trace of a focus will yield the generating curve of a CMC surface and that all axially symmetric CMC surfaces arise in this way. The correspondence is as follows: circle  $\leftrightarrow$  cylinder, ellipse  $\leftrightarrow$  unduloid, hyperbola  $\leftrightarrow$  nodoid, parabola  $\leftrightarrow$  catenoid. The sphere can be obtained as a limiting case of unduloids. The integration of the equation for the rolling curve is reminiscent of another important appearance of the conic sections; Kepler's integration of the equations of motion of an object subjected to a central force field. It is likely that Delaunay, who was an astronomer by profession and who had written his dissertation on variational calculus, was influenced by Kepler's calculations.

Now consider an axially symmetric anisotropic surface energy functional  $F$ . This is equivalent to the Wulff shape  $W$  being axially symmetric and we let  $(u(v), v)$  denote the generating curve of  $W$ . Let  $\mu_1$ , (respectively  $\mu_2$ ) denote the principal curvature of  $W$  along a meridian, (respectively a parallel), with respect to the inward pointing surface normal of  $W$ . Likewise, we let  $\kappa_i$  denote the corresponding principal curvatures of an axially symmetric surface  $\Sigma$ , with respect to the *outward* pointing surface normal. In this case, the equilibrium equation reduces to:

$$\frac{\kappa_1}{\mu_1} + \frac{\kappa_2}{\mu_2} \equiv 2\Lambda_0 = \text{constant}$$

which is a special case of Eq. (2).



**Figure 1.** Generating curves of  $\Sigma$  and  $W$ .

The axially symmetric surfaces with constant anisotropic mean curvature will be called anisotropic Delaunay surfaces [7]. These can be found as follows. Let  $(r,z)$  denote the generating curve of such a surface  $\Sigma$ . To each point on  $\Sigma$ , there is a unique point on  $W$  where the normals to  $\Sigma$  and  $W$  agree.

Then at these points,  $u$  and  $r$  are related by the quadratic equation (see Fig. 1):

$$\frac{2ru}{\mu_2} + \Lambda r^2 = c \quad (4)$$

where  $c$  is a real constant and  $\Lambda$  is the anisotropic mean curvature. Once  $r = r(u, \Lambda, c)$  is determined from Eq. (4), the vertical coordinate can be determined by:

$$z = \int_{v_0}^v r_u(u(\hat{v})) d\hat{v}$$

To generate a wide variety of examples, we employ the Gielis formula [8]:

$$r = r(\theta, m, n_1, n_2, n_3, a, b) = \left\{ \left| (1/a) \cos \left( \frac{m\theta}{4} \right) \right|^{n_2} + \left| (1/b) \sin \left( \frac{m\theta}{4} \right) \right|^{n_3} \right\}^{-n_1} \quad (5)$$

This formula is an excellent source of curves with prescribed symmetries yielding a curve in polar form. Its utility in modeling natural shapes in a wide variety of circumstances is reviewed in [9]. We refer to the polar graphs given by Eq. (5) as *Gielis curves*. Given two *convex* Gielis curves,  $(u(\sigma), v(\sigma)), (\alpha(t), \beta(t))$  we form a “product” Wulff shape parameterized by:

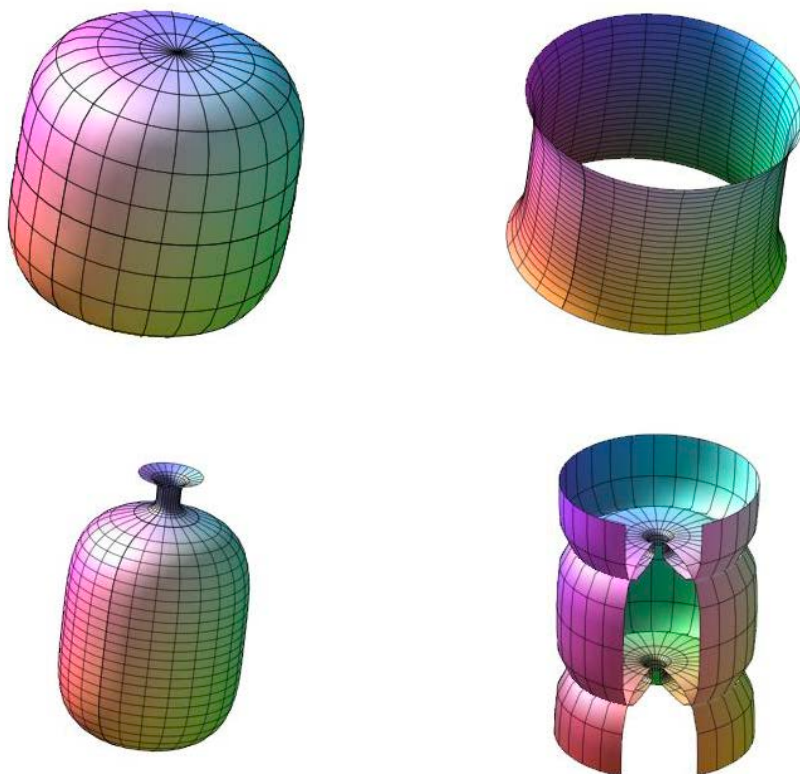
$$\chi(\sigma, t) = (u(\sigma)[\alpha(t), \beta(t)], v(\sigma)) \quad 0 \leq \sigma \leq \bar{\sigma}, 0 \leq t \leq \bar{t} \quad (6)$$

For this type of Wulff shape, the analogues of the constant anisotropic surfaces found above can be found by replacing the cross-sectional circles by the  $(\alpha, \beta)$  curve as in Fig. 2 and Fig. 3 (see [10]).

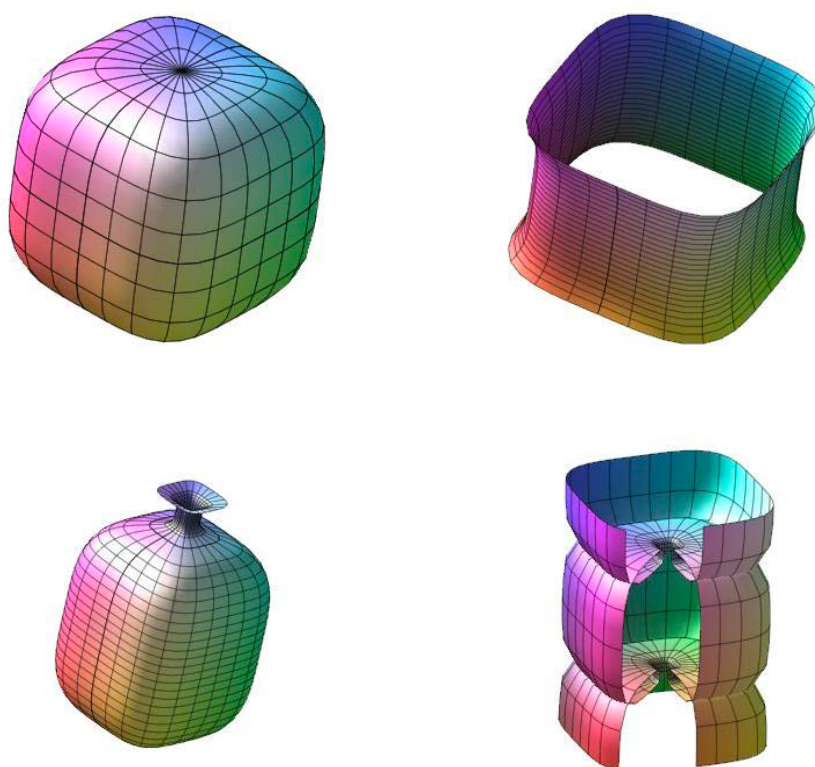
There is at least one earlier appearance of a type of anisotropic Delaunay surface in the literature. Bernal and Fankuchen observed that equilibrium droplets of a type of nematic liquid crystal which was extracted from tobacco plant virus, take a form which they refer to as a tactoid, which resembles an American football. They state that the surface of these shapes satisfies Eq. (2).

### 3. HELICOIDAL SURFACES

Oscar Perdomo [11] gave a remarkable generalization of the rolling construction which produces constant mean curvature surfaces which are invariant under a helicoidal motion. Such surfaces are known as CMC twizzlers.



**Figure 2.** Upper left: Wulff shape. Upper right: catenoid. Bottom left: unduloid. Bottom right: nodoid.



**Figure 3.** Upper left: Wulff shape. Upper right: catenoid. Bottom left: unduloid. Bottom right: nodoid.

The immersion is represented as the orbit of a planar generating curve  $(x(s), y(s))$  under the helicoidal motion, so that one has:

$$X(s, \vartheta) = \left( (x(s) + iy(s))e^{-i\omega\vartheta}, \vartheta + c \right) \quad (7)$$

where the angle  $\omega$  is a constant which determines the pitch of the twizzler. To find the generating curve, Perdomo introduces a curve called the treadmill sled. The generating curve is the trace of a point which is rigidly attached to the treadmill sled which rolls without slipping on a treadmill. The coordinates  $(\xi_1, \xi_2)$  of the treadmill sled are determined from the conservation law:

$$\Lambda(\eta_1^2 + \eta_2^2) + \frac{2\eta_2}{\sqrt{1 + \omega^2\eta_1^2}} + A \equiv 0 \quad (8)$$

where  $A$  is a real constant. The generating curve is related to the treadmill sled by:

$$x + iy = -(\eta_1 + i\eta_2) \exp\left(-i \int \frac{d\eta_2}{\eta_1}\right) \quad (9)$$

where  $\eta_2 = \eta_2(\eta_1)$  has been determined from Eq. (8).

Perdomo's construction generalizes to the anisotropic cases provided that the Wulff shape  $W$  is smooth and axially symmetric [12] (see Fig. 4, Fig. 5 and Fig. 6). The only modification in the construction is that Eq. (8) is changed to:

$$\Lambda(\eta_1^2 + \eta_2^2) + \frac{2\eta_2}{\mu_2(\omega\eta_1/\sqrt{1 + \omega^2\eta_1^2})\sqrt{1 + \omega^2\eta_1^2}} + A \equiv 0 \quad (10)$$

It is worth noting that if a helicoidal or axially symmetric surface is represented in non-parametric form  $z = z(r) + \lambda\theta$ , then both Eq. (4) and Eq. (10) can be expressed as the single equation:

$$\frac{2rz_r v_3}{\mu_2(v_3)} + \Lambda r^2 = \text{constant} \quad (11)$$

## 4. ROTATING ANISOTROPIC DROPS

We will consider the equilibrium shape of a non-liquid drop rotating with a constant angular velocity  $\Omega$  about a vertical axis. The surface of the drop, which we denote by  $\Sigma$ , is represented as a smooth surface. The bulk of the drop is assumed to be occupied by an incompressible liquid of a constant mass density  $\rho_1$  while the drop is surrounded by a fluid of constant mass density  $\rho_2$ . The drop's free surface energy is assumed to be given by an anisotropic surface energy  $\mathcal{F}$ . The rotation contributes a second energy term of the form  $-\Omega^2\Delta\mathcal{I}$ , where  $\Delta\mathcal{I}$  is difference of moments of inertia about the vertical axis:

$$\Delta\mathcal{I} = (\rho_1 - \rho_2) \int_U R^2 dv$$

where  $R = \sqrt{x^2 + y^2}$ . The last term represents twice the rotational kinetic energy of the drop.

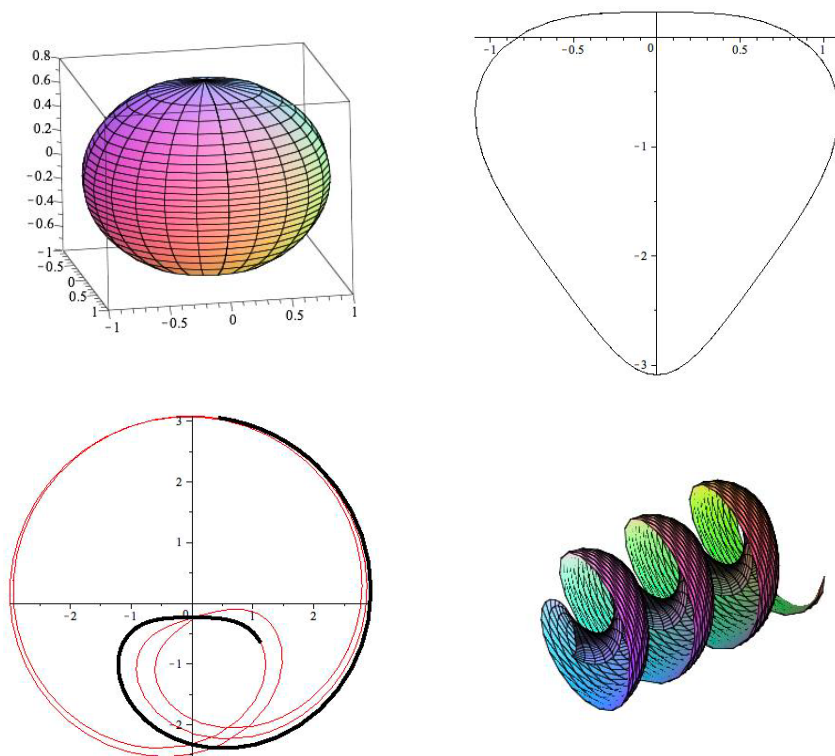
The total energy is thus of the form:

$$\mathcal{E} = \mathcal{F} - \frac{\Omega^2}{2} \Delta\mathcal{I} + \Lambda_0 \mathcal{V} \quad (12)$$

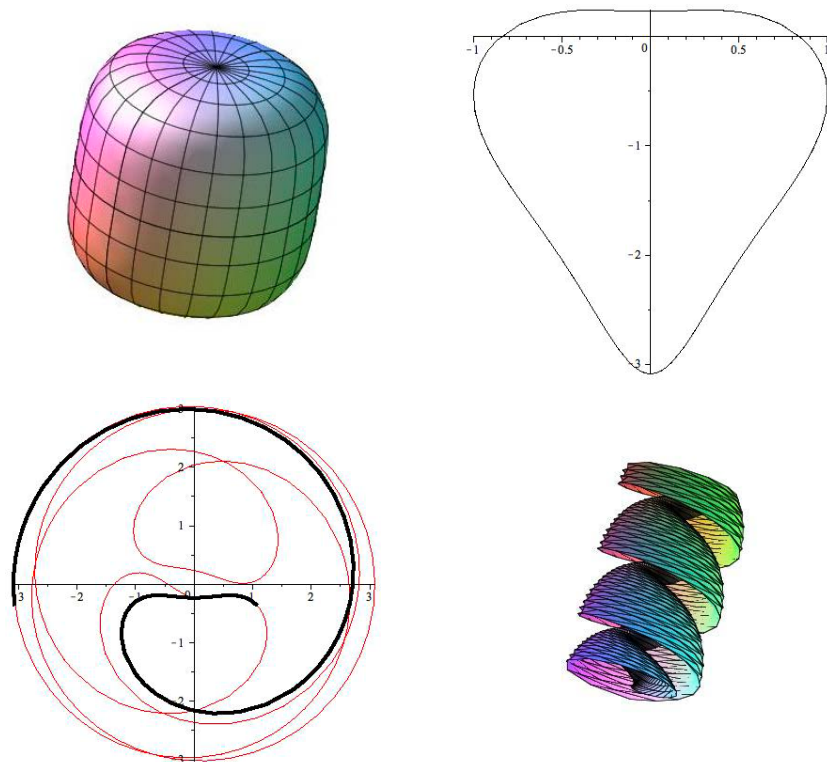
where  $\mathcal{V}$  denotes the volume of the drop and  $\Lambda_0$  is a Lagrange multiplier. Let  $\Delta\rho = \rho_1 - \rho_2$ , then by introducing a constant  $a = (\Delta\rho)\Omega^2$ , we can write the functional in the form:

$$\mathcal{E}_{a, \Lambda_0} = \mathcal{F} - \frac{a}{2} \int_U R^2 dV + \Lambda_0 \mathcal{V}$$

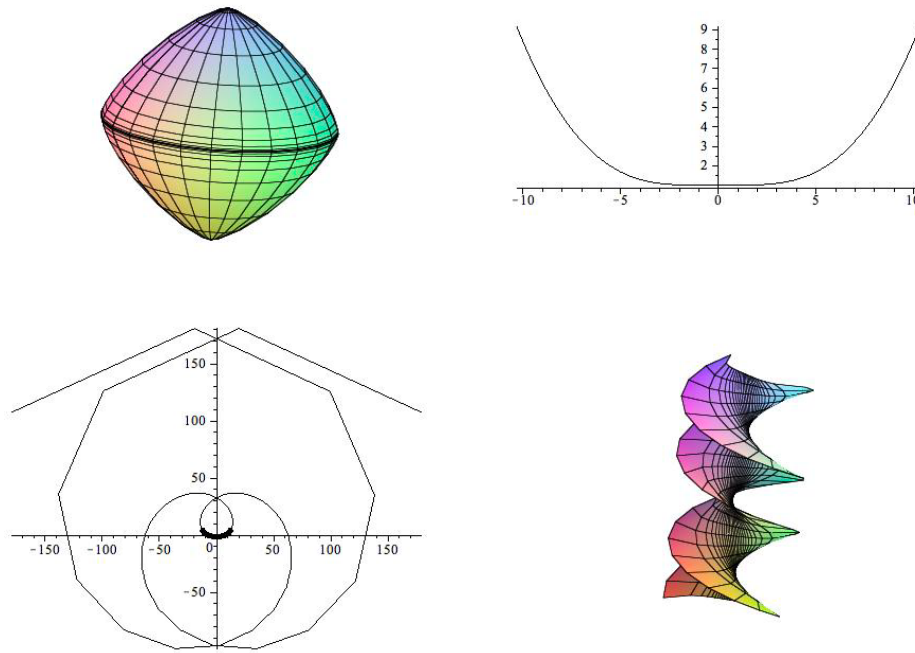




**Figure 4.** Upper left: Wulff shape for  $\gamma = 1 - 0.2v_3^2$ . Upper right: treadmill sled  $(\Lambda, A, \omega) = (-.7, .5, 2)$ . Bottom: Corresponding generating curve and twizzler surface. The darkened arc of the generating curve corresponds to the displayed portion of the surface.



**Figure 5.** Upper left: Wulff shape for  $\gamma = 1 + 0.2v_3^2$ . Upper right: treadmill sled  $(\Lambda, A, \omega) = (-.7, .5, 2)$ . Bottom: Corresponding generating curve and twizzler surface. The darkened arc of the generating curve corresponds to the displayed portion of the surface.



**Figure 6.** Upper left: Wulff shape for  $\gamma = ((1 - v_3^2)^2 + v_3^4)^{1/4}$ . Upper right: treadmill sled  $(\Lambda, A, \omega) = (0, 2, 0.2)$ . Bottom: Corresponding generating curve and twizzler surface. The darkened arc of the generating curve corresponds to the displayed portion of the surface.

where  $U$  is the three-dimensional region occupied by the bulk of the drop. The Euler-Lagrange equation satisfied at points on the surface of the drop is:

$$\Lambda = \Lambda_0 - \frac{aR^2}{2} \quad (13)$$

In the special case  $\gamma = \gamma(v_3)$  if the surface is given as an axially symmetric graph  $x_3 = Z(r)$  then:

$$\Lambda = \frac{-1}{r} \left( \frac{rv_3}{\mu_2} Z_r \right)_r$$

where  $\mu_2 = (\gamma - v_3 \gamma'(v_3))^{-1}$  is the principal curvature of  $W$  along a meridian. Using this, we obtain the first integral:

$$\frac{2rv_3}{\mu_2} Z_r + \Lambda_0 r^2 - \frac{\alpha r^4}{4} = C \quad (14)$$

where  $C$  is a constant of integration. If  $(u, v)$  denote the radial and vertical coordinates of  $W$  at a point where the normals to  $\Sigma$  and  $W$  agree, then this equation can be expressed as:

$$2ur + \Lambda_0 r^2 - \frac{\alpha r^4}{4} = \text{constant} \quad (15)$$

To see this, we note that the principal curvatures of  $W$  with respect to the inward pointing normal are given by  $\mu_i, i = 1, 2$  where  $1/\mu_2 = \gamma - v_3 \gamma'(v_3), \gamma_1 = (1 - v_3^2) \gamma''(v_3) + 1/\mu_2$ . Then  $u = \sqrt{1 - v_3^2}/\mu_2, v = v_3/\mu_2 + \gamma'(v_3)$  parameterizes the generating curve of  $W$ . Using  $v_3 = (1 + Z_r^2)^{-1/2}$ , it follows that  $u = Z_r v_3/\mu_2$ .

For closed genus zero surfaces we can let  $r$  approach 0 and we obtain that  $C = 0$  and Eq. (15) reduces to a cubic which we write as:

$$\frac{\alpha r^3}{4} - \Lambda_0 r - 2u = 0 \quad (16)$$

The discriminant is  $\vartheta(v_3) = a\Lambda_0^3 - 27(a^2/4)(u(v_3))^2$ . We first assume  $0 < a = (\Delta\rho)\Omega^2$ . Then  $\Lambda_0 \leq 0$  implies that the discriminant is always negative and the cubic has a unique real root. The surface can be found explicitly by setting  $k := \sqrt{-16\Lambda_0/(3a)}$  using:

$$r = k \sinh \left( \frac{1}{3} \operatorname{arcSinh} \left( \frac{32u}{ak^3} \right) \right)$$

Surfaces of this type are shown in [Fig. 7](#) and [Fig. 8](#).

$$\frac{ar^3}{4} - \Lambda_0 r - 2u = 0 \quad (17)$$

We will describe a construction of the part of the generating curve  $C$  which lies in the first quadrant. If  $W$  is invariant by a reflection through the plane  $x_3 = 0$ , then  $C$  will also be invariant under a horizontal reflection. If  $0 < \Lambda_0 < 3(au_{\max}^2/4)^{1/3}$  the discriminant has a sign change when:

$$u_0 = 2a \left( \frac{\Lambda_0}{3a} \right)^{3/2}$$

In this case,  $C$  will be obtained by piecing together three curves. Let:

$$k := \sqrt{\frac{16\Lambda_0}{3a}}$$

For  $u_{\max} \geq u \geq u_0$ , the cubic has a unique real root:

$$r_1 = k \cosh \left( \frac{1}{3} \operatorname{arcCosh} \left( \frac{32u}{ak^3} \right) \right)$$

For  $u_0 \geq u \geq -u_0$ :

$$r_2 = k \cos \left( \frac{1}{3} \operatorname{arcCos} \left( \frac{32u}{ak^3} \right) \right)$$

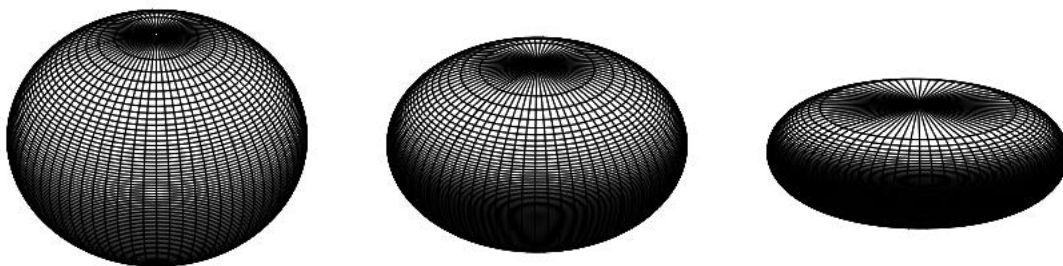
For  $0 \geq u \geq -u_0$ :

$$r_3 = k \cos \left( \frac{1}{3} \operatorname{arcCos} \left( \frac{32u}{ak^3} \right) - \frac{2\pi}{3} \right)$$

In all three cases, the vertical coordinate of the generating curve  $(r, Z)$  is obtained from:

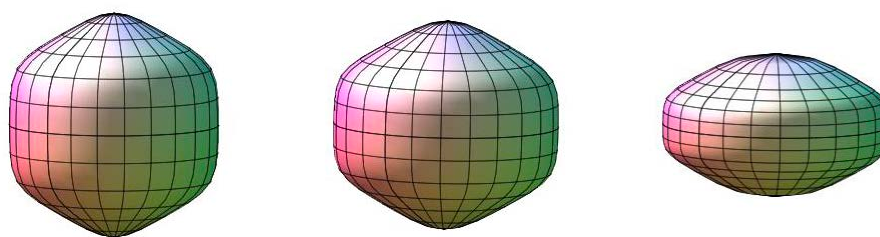
$$Z_i(v) = \int_{v_0}^v \partial_u(r_i)(u(\hat{v}))d\hat{v} + C_i, i = 1, 2, 3$$

for appropriate constants  $C_i$  which must be chosen so that the three curves fit together as in [Fig. 9](#).

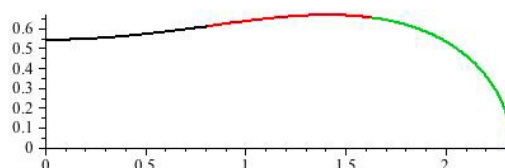


**Figure 7.** Equilibrium drops for the functional  $\gamma = 1 - 0.2v_3^2$ . Left:  $(\Lambda_0, a) = (0.2, 0)$ . Center:  $(\Lambda_0, a) = (-1, 1)$ . Right:  $(\Lambda_0, a) = (0, 1)$ .





**Figure 8.** Equilibrium drops. Left:  $a = 0$ . Center:  $a = 1$ . Right:  $a = 100$ .



**Figure 9.** Gluing branches of the profile curve.

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